Expansions and Extensions

Ergodic, combinatorial and geometric properties
of β-expansions with arbitrary digits

Thesis committee:
Prof. dr. F. M. Dekking, Technische Universiteit Delft (DIAM)
Prof. dr. F. den Hollander, Universiteit Leiden
Prof. dr. V. Komornik, Université de Strasbourg
Prof. dr. B. Solomyak, University of Washington
Dr. W. Steiner, Université Paris Diderot - Paris 7

Cover: On the front, from top to bottom, left to right, $\widehat{X}$ from Example 5.3.6, the tiling from Figure 5.11, Figure 4.6 and Figure 4.7. On the back, transformation U from Section 3.3.3.

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Expansions and Extensions

Ergodic, combinatorial and geometric properties of β-expansions with arbitrary digits

Ontwikkelen en Uitbreiden

Ergodische, combinatorische en meetkundige eigenschappen van β-expansies met willekeurige cijfers

(met een samenvatting in het Nederlands)

Proefschrift

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Charlene Catharina Cornelia Johanna Kalle

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Promotor:	Prof. dr. R. D. Gill
Co-promotor:	Dr. K. Dajani

Zo als een vogel in de stille nacht
Op ééns ontwaakt, omdat de hemel gloeit,
En denkt, 't is dag en heft het kopje en fluit,

Maar eer 't zijn vaakrige oogjes gans ontsluit,
Is het weer donker, en slechts droevig vloeit
Door 't sluimerend geblaarte een zwakke klacht.

— Willem Kloos, Ik ween om bloemen

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Expand your horizon

Real numbers can be represented in many different ways. The one that we are most familiar with is the decimal expansion. If we read 0.25 somewhere, we usually interpret this as

$$0.25 = \frac{2}{10} + \frac{5}{100}.$$

Any number between 0 and 1 can be written in such a way, that is as a sum of fractions with powers of 10 as denominators and integers between 0 and 9 in the numerator. To be more precise, each $x \in [0, 1]$ can be written as

$$x = \sum_{k=1}^{\infty} \frac{b_k}{10^k},$$

where $b_k \in \{0, \dots, 9\}$ for all $k \geq 1$. This expression is called the decimal expansion of x. There is nothing special about the number 10 here. We could just as well use powers of 2 in the denominators and 0's and 1's in the numerators. What we get then are the binary expansions. In fact, we can use any integer $r > 1$ to represent numbers in the interval $[0, 1]$ by an infinite sum of fractions with powers of r in the denominators and elements from the set $\{0, \dots, r-1\}$ in the numerators. The expansion is then called the r-adic expansion. The number r is called the *base* of the expansion and the numbers $0, 1, \dots, r-1$ are called the *digits*.

There is more. In 1957, Rényi was the first to introduce β-expansions. These expansions have a non-integer $\beta > 1$ as a base and use the integers between 0 and $\lfloor \beta \rfloor$ as digits, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Although at first sight the difference between an integer base and a non-integer base might not seem so big, the expansions have very different properties. One big difference is that the r-adic expansions are basically unique, while for every non-integer base β, almost all numbers have infinitely many different expansions in that base. The properties of β-expansions have been studied from various points of view. We will give a small overview in Section 2.1. Recently, Daubechies et al. used expansions with the golden ratio as a base to improve analog-to-digital converters (see [DGWY]).

We can go even further. A next logical way of generalizing is by taking more general sets of digits. Notice that the β-expansions introduced by Rényi, use digit sets containing only integers and only the integers between 0 and $\lfloor \beta \rfloor$. The main topic of this thesis is describing the properties of β-expansions that use an arbitrary set of real numbers as a digit set. A β-expansion of a number x will be any expression of the form

$$x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k},$$

where $\beta > 1$ is a real number and all the digits b_k are in some finite set of real numbers $A = \{a_0, a_1, \dots, a_m\} \subset \mathbb{R}$. We will give procedures to obtain such expansions and study their properties. These procedures will be of a dynamical nature, which means that we will introduce transformations that generate the expansions by iteration. This makes it possible to use ergodic theory as a tool and study the expansions through the dynamical systems that generate them.

The expansions can be identified with the sequence of digits $b_1 b_2 \cdots$. If one knows this sequence and the corresponding base, then one knows the number that the sequence represents. On the other hand, if one knows the number, then there are ways to obtain an expansion in a certain base and with certain digits and thus also to obtain the corresponding sequence of digits. As a consequence, when we study the expansions the set of digit sequences is always present in the background and results about expansions can often be translated to results about digit sequences and vice versa.

The main emphasis in this dissertation is on the transformations that generate the β-expansions and on their measure theoretical and ergodic properties. In general, there is not one way to obtain expansions dynamically, but there are many different ways. This is not the case for the r-adic expansions, where there is basically just one way to obtain them. In the next section we will look at the r-adic expansions in detail. They can serve as an easy point of reference for the following chapters. We will give a number of their characteristic properties and use this to introduce the main concepts of this dissertation.

1.2 r-adic expansions and transformations

1.2.1 *r-adic expansions and the dynamical system*

How do we get r-adic expansions for an integer $r > 1$? Let x be any real number in the interval $[0, 1]$. We are looking for an expression of the form

$$x = \sum_{k=1}^{\infty} \frac{b_k}{r^k},$$

with $b_k \in \{0, 1, \ldots, r-1\}$ for all $k \geq 1$. Such an expression is called an *r-adic expansion of x*. Note that the smallest number we can get in this way is by taking all the b_i's equal to 0. This gives us 0. The largest number we can get is by taking all b_k's equal to $r-1$. This gives 1. We can obtain an r-adic expansion of x by using a recursive greedy algorithm. Suppose we already know what $b_1, \ldots, b_{n-1}$ are. Then we can take b_n to be the largest integer between 0 and $r-1$, such that

$$\frac{b_1}{r} + \cdots + \frac{b_{n-1}}{r^{n-1}} + \frac{b_n}{r^n} \leq x. \tag{1.1}$$

Since the sum on the left hand side gets bigger and bigger, the difference between this sum and x gets smaller and smaller and in the limit it equals x. The algorithm is called greedy since it takes the largest digit possible at each step. Another way of getting an r-adic expansion of x is by using a transformation. This transformation is defined by $T_r x = rx \pmod 1$ for all $x \in [0, 1)$ and $T_r 1 = 1$. The transformation is shown in Figure 1.1 for $r = 3$. The digits b_k are determined by the iterations of the transformation

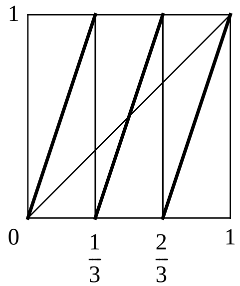


Figure 1.1 The transformation T_3.

T_r. The transformation divides the interval $[0, 1)$ into r pieces, the intervals $\left[\frac{i}{r}, \frac{i+1}{r}\right)$ for $0 \leq i \leq r-1$. To each of these intervals we assign a digit. The interval $\left[\frac{i}{r}, \frac{i+1}{r}\right)$ corresponds to the digit i. The first digit of the expansion of an x, b_1, is determined by the interval in which x lies. Then we apply the transformation T_r to x. We let the position of $T_r x$ determine the second digit b_2 and continue in this manner. In Figure 1.2 we see how we get the first five digits of the expansion of $\frac{1}{2}\sqrt{3}$ in base 3 in this way. We can make this

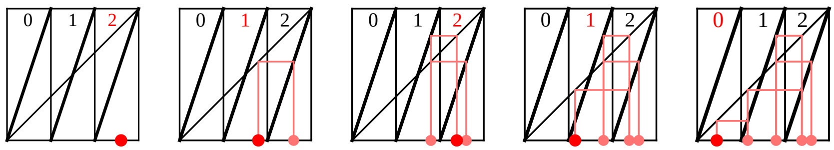


Figure 1.2 The first five digits of the 3-adic expansion of $\frac{1}{2}\sqrt{3}$ are 21210.

procedure more precise in the following way. For each $x \in [0, 1)$, define the sequence of digits by setting $b_1 = b_1(x) = i$, if $x \in \left[\frac{i}{r}, \frac{i+1}{r}\right)$ and for each $k \geq 1$, set $b_k = b_k(x) = b_1(T_r^{k-1}x)$. Then we can write $T_r x = rx - b_1$. Inverting this relation gives $x = \frac{b_1}{r} + \frac{T_r x}{r}$. If we repeat this, after n steps we get

$$x = \frac{b_1}{r} + \frac{T_r x}{r} = \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{T_r^2 x}{r^2} = \cdots = \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots + \frac{b_n}{r^n} + \frac{T_r^n x}{r^n}.$$

Since $T_r^n x \in [0,1)$ for all $x \in [0,1)$, this sum converges and gives an r-adic expansion of x. If we set $b_1(1) = r-1$ and $b_k(1) = b_1(T_r^{k-1}1)$, then $b_k(1) = r-1$ for all $k \geq 1$. So, T_r generates digit sequences $b(x) = (b_k(x))_{k\geq 1}$ for each $x \in [0,1]$ by iterations. By these definitions the sequence $b(1)$ is the only sequence generated by T_r that ends in an infinite string of $(r-1)$'s.

1.2.2 *r-adic expansions and sequences*

The sequence $b(x) = (b_k(x))_{k\geq 1}$ that corresponds to the expansion of x in base r, as given by T_r, is called the *digit sequence of x generated by T_r*. We usually write $b_1 b_2 \cdots$ instead of $(b_k(x))_{k\geq 1}$. For sequences, we have the following notations. Let A be a finite set of real numbers. Then A^ω denotes the set of right-sided sequences of elements in A, i.e., $b \in A^\omega$ if and only if $b = b_1 b_2 \cdots$ with $b_i \in A$ for all $i \geq 1$. For all $n \geq 1$, A^n is the set of all finite sequences of length n of elements from A, i.e., $b \in A^n$ means that $b = b_1 \cdots b_n$ with $b_i \in A$ for all $1 \leq i \leq n$. The set ${}^\omega A$ denotes the set of left-sided sequences of elements in A and $A^{\mathbb{Z}}$ denotes the set of two-sided sequences. Usually, when we write $A^{\mathbb{Z}}$, then the 0-th position has some importance. We use $(b_k \cdots b_m)^\omega$ to denote a periodic block of digits $b_k \cdots b_m b_k \cdots b_m b_k \cdots$ and similarly ${}^\omega(b_k \cdots b_m)$ denotes the block $\cdots b_m b_k \cdots b_m b_k \cdots b_m$. For each $n \geq 1$, $(b_k \cdots b_m)^n$ means

$$\underbrace{(b_k \cdots b_m)(b_k \cdots b_m) \cdots (b_k \cdots b_m)}_{n \text{ times}}.$$

For two sequences $w = (w_k)_{k\leq 0} \in {}^\omega A$ and $u = (u_k)_{k\geq 1} \in A^\omega$, we write $w \cdot u$ for the two-sided sequence $\cdots w_{-1} w_0 u_1 u_2 \cdots \in A^{\mathbb{Z}}$.

Definition 1.2.1. A sequence $b_1 b_2 \cdots \in A^\omega$ is called *purely periodic* if there is a $p \geq 1$ such that $b_1 b_2 \cdots = (b_1 \cdots b_p)^\omega$. It is called *eventually periodic* if there are numbers $m, p \geq 1$, such that $b_1 b_2 \cdots = b_1 \cdots b_m (b_{m+1} \cdots b_{m+p})^\omega$. Similar definitions can be given for sequences in ${}^\omega A$.

Let $x \in [0,1]$. If the expansion of x, generated by T_r, is eventually periodic, then $x \in \mathbb{Q}$. On a set of sequences, we let $\prec$ denote the lexicographical ordering.

Definition 1.2.2. Consider the set of right-sided sequences A^ω. We use $\prec$ to denote the *lexicographical ordering* on this set, i.e., for two sequences $u, u' \in A^\omega$, we have $u_1 u_2 \cdots \prec u'_1 u'_2 \cdots$ if and only if $u \neq u'$ and for the first index k such that $u_k \neq u'_k$, we have $u_k < u'_k$. We can extend this definition in the obvious way to define $\preceq$, $\succ$ and $\succeq$.

We can consider the set of sequences A^ω as a topological space under the product topology. A subbase of the product topology is given by the cylinder sets of length one.

Definition 1.2.3. The sets

$$\{u = (u_k)_{k\geq 1} \in A^\omega \mid u_1 \cdots u_n = b_1 \cdots b_n, \text{ with } b_j \in A \text{ for } 1 \leq j \leq n\}$$

are called *cylinder sets of length n*.

The cylinder sets are compact in the product topology. We can also turn A^ω into a probability space. Let $\mathcal{A}$ be the σ-algebra on A^ω generated by the cylinder sets. On the measurable space $(A^\omega, \mathcal{A})$, we consider a product measure: if $A = \{a_0, \ldots, a_m\}$ and $p = (p_0, \ldots, p_m)$ is a probability vector, then the product measure μ_p on $(A^\omega, \mathcal{A})$ is given for the cylinder sets by

$$\mu_p(\{u = (u_k)_{k\geq 1} \in A^\omega \mid u_1 \cdots u_n = b_1 \cdots b_n\}) = p_{b_1} \cdots p_{b_n}.$$

Let σ denote the left-shift on A^ω, i.e., for a sequence $u = (u_k)_{k\geq 1} \in A^\omega$ we set $\sigma(u) = (u_k)_{k\geq 2}$.

Definition 1.2.4. The system $(A^\omega, \mathcal{A}, \mu_p, \sigma)$ is called the *left-sided Bernoulli shift* or *one-sided Bernoulli shift* on the symbols $a_0, \ldots, a_m$. We can naturally extend this definition from A^ω to $A^\mathbb{Z}$ to get the *two-sided Bernoulli shift*. The system is called the *uniform Bernoulli shift* if $p = \left(\frac{1}{m+1}, \ldots, \frac{1}{m+1}\right)$. The measure μ_p is called the *uniform product measure*.

Let $x = \sum_{k=1}^\infty \frac{b_k}{r^k}$, where the b_k's are obtained from the iterations of the transformation T_r. Then

$$T_r x = rx - b_1 = \sum_{k=1}^\infty \frac{b_{k+1}}{r^k}.$$

So, if $b(x) = b_1(x)b_2(x)\cdots$ is the digit sequence of x generated by T_r, then $b(T_r x) = b_2(x)b_3(x)\cdots$. In this sense, T_r behaves like a left-shift on the set of digit sequences it generates. Using this, we get that

$$\begin{aligned} b_n = i \quad &\Leftrightarrow \quad \frac{i}{r} \leq T_r^{n-1}x < \frac{i+1}{r} \Leftrightarrow \frac{i}{r^n} \leq \sum_{k=1}^\infty \frac{b_{k+n-1}}{r^{k+n-1}} < \frac{i+1}{r^n} \\ &\Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{b_k}{r^k} + \frac{i}{r^n} \leq \sum_{k=1}^\infty \frac{b_k}{r^k} < \sum_{k=1}^{n-1} \frac{b_k}{r^k} + \frac{i+1}{r^n}. \end{aligned}$$

Thus, the expansion that T_r generates by iteration is equal to the expansion that the greedy algorithm from (1.1) produces. This leads to the question: can an $x \in [0,1)$ have other r-adic expansions?

Looking at the transformation we can notice the following. If at any point we use any other digit than the digit given by the transformation, then the result becomes either too large or too small. To be more precise, suppose $x \in \left(\frac{i}{r}, \frac{i+1}{r}\right)$ and suppose that we want to write $x = \sum_{k=1}^\infty \frac{b_k}{r^k}$ with $b_k \in \{0, \ldots, r-1\}$. If we take $b_1 > i$, then

$$\sum_{k=1}^\infty \frac{b_k}{r^k} \geq \frac{i+1}{r} + \sum_{k=2}^\infty \frac{0}{r^k} > x.$$

In the same way, if we take $b_1 < i$, then

$$\sum_{k=1}^\infty \frac{b_k}{r^k} \leq \frac{i-1}{r} + \sum_{k=2}^\infty \frac{r-1}{r^k} \leq \frac{i-1}{r} + \frac{r-1}{r^2-r} = \frac{i}{r} < x.$$

If this happens, we cannot repeat the same algorithm, or we cannot iterate the transformation. So, in fact, as long as $T_r^n x \notin \left\{\frac{i}{r} \,|\, 1 \le i \le r-1\right\}$ for all n, the digits given by the transformation T_r are unique. If $x = \frac{i}{r}$ for some $1 \le i \le r-1$, then there are two possibilities. Either $b_1 = i$ and $b_k = 0$ for all $k \ge 2$ or $b_1 = i-1$ and $b_k = r-1$ for all $k \ge 2$. These x's thus have exactly two possible expansions. The same holds for all numbers that are eventually mapped on a point $\frac{i}{r}$ by T_r. These are the numbers $\frac{k}{r^n}$ for $0 < k \le r^n - 1$ and $n > 1$. Hence, almost all numbers in $[0,1]$ have a unique r-adic expansion and the numbers that do not have a unique expansion, have exactly two expansions. Note that for points with two possible expansions, T_r only generates the one which ends in an infinite string of 0's. This is, because in the definition of T_r, we choose the intervals to be left-closed and right-open.

Now, consider the set of digit sequences that T_r produces. If we look at the transformation, we see that at each step each digit can be followed by each other digit. So, from the transformation it is clear that the set of digit sequences generated by T_r is just the set of all possible sequences of elements from $\{0,\dots,r-1\}$, without those sequences that end in an infinite string of $(r-1)$'s.

1.2.3 *r-adic transformations and ergodic theory*

The transformation T_r is defined on the interval $[0,1]$. Let $\mathcal{B}$ and $\mathcal{L}$ denote the Borel and Lebesgue σ-algebra on $[0,1]$ respectively. We will use this notation throughout the text. Most of the time we will consider them on the interval $[0,1)$ instead of $[0,1]$, but it will always be clear which interval is meant. The big advantage of having a transformation to generate the number expansions, is that we can use ergodic theory as a tool. This is, provided that we have an invariant measure for the transformation. With an invariant measure, we mean the following.

Definition 1.2.5. If $(X, \mathcal{F}, \mu)$ is a probability space and $T : X \to X$ a transformation, then the measure μ is said to be *T-invariant* if for each set $B \in \mathcal{F}$, $\mu(B) = \mu(T^{-1}B)$. We also say that T is *measure preserving* or *invariant* with respect to μ. The quadruple $(X, \mathcal{F}, \mu, T)$ is called a *dynamical system*.

It is enough to check this property for a generating semi-algebra. Let λ denote the one-dimensional Lebesgue measure. Then it is easily seen that λ is T_r-invariant: for each interval $[a,b) \subseteq [0,1)$, the inverse image of $[a,b)$ under T_r is the set

$$T_r^{-1}[a,b) = \bigcup_{i=0}^{r-1} \left[\frac{a+i}{r}, \frac{b+i}{r}\right),$$

where the union on the right is disjoint. Hence,

$$\lambda([a,b)) = b - a = \sum_{i=0}^{r-1} \lambda\left(\left[\frac{a+i}{r}, \frac{b+i}{r}\right)\right) = \lambda\big(T_r^{-1}[a,b)\big).$$

Thus, the quadruple $([0,1], \mathcal{B}, \lambda, T_r)$ is a dynamical system.

In general, finding an invariant measure for a transformation is not a simple task. For a Lebesgue measurable transformation $T : [0,1] \to [0,1]$ that is piecewise linear, expanding, we can do the following. If h is a non-negative, integrable function, then h is a density of a T-invariant measure, absolutely continuous with respect to the Lebesgue measure, if h satisfies the *Perron-Frobenius equation*:

$$h(x) = \sum_{y \in T^{-1}\{x\}} h(y)\,|T'y|^{-1} \quad \lambda \text{ a.e.} \tag{1.2}$$

Luckily, there are many results on invariant measures for piecewise linear, expanding maps. In Section 2.3 we encounter results from Lasota and Yorke ([LY73]) and Li and Yorke ([LY78]) on the existence of invariant measures for such maps. There are also some results on formulas for densities of these invariant measures, that are absolutely continuous with respect to the Lebesgue measure. In particular, Kopf ([Kop90]) considered a class of piecewise linear, expanding maps from the interval $[0,1]$ to itself, that leave the points 0 and 1 fixed. He constructed a matrix M, the entries of which consist of infinite sums of indicator functions, and he used a vector from the nullspace of M to obtain the density function. A more recent result can be found in [Gór09] from Góra. He considered an even more general class of piecewise linear maps. In his setting, the maps only have to be eventually expanding, which means that for each slope β_k there must exist an $n \geq 1$ such that $|\beta_k|^n > 1$. The slopes can also be negative, under the same condition. For this class of transformations, Góra constructed a matrix S and used the solutions of a certain linear system involving this S to obtain the density function. Two main differences between their two methods are the following. First of all, Kopf makes the extra assumption that the points 0 and 1 are fixed. More importantly, Kopf obtains all invariant densities, while Góra gives only one version of the density for each ergodic component. However, both these matrices M and S are difficult to obtain.

For measure preserving transformations, we have the following famous result.

Theorem 1.2.1 (Poincaré Recurrence Theorem)**.** *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system and $B \in \mathcal{F}$ with $\mu(B) > 0$. Then for μ a.e. $x \in B$ there is an $n \geq 1$, such that $T^n x \in B$.*

This implies that almost every point from B will eventually return to B. An immediate consequence is that almost all elements from B will return to B infinitely often. Since T_r is invariant with respect to λ, for each interval $\left[\frac{i}{r}, \frac{i+1}{r}\right)$, and almost every element x from this interval, $T_r^n x$ will again be in this interval for infinitely many n. This implies that the digit i will occur on infinitely many places in the expansion of x.

The Poincaré Recurrence Theorem is concerned with the set of points $\{T^n x\}_{n \geq 0}$. This set is called the orbit of x under T.

Definition 1.2.6. If $T : X \to X$ is a transformation, then for each $x \in X$, the set $\{T^n x\}_{n \geq 0}$ is called the *orbit of x under T*. We say that the orbit of x *hits* a certain set if there is an element of the orbit in that set.

To apply ergodic theory, we would like to have a transformation that is ergodic. For the transformation T_r this holds.

Definition 1.2.7. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. Then the transformation T is *ergodic* with respect to μ if for all $B \in \mathcal{F}$ with $B = T^{-1}B$, we have $\mu(B) = 0$ or 1.

The proof that T_r is *ergodic* with respect to λ is done, using Knopp's Lemma. We will not give the proof here, but refer the reader to textbooks as [DK02a] by Dajani and Kraaikamp for example. The properties of T_r allow us to use the Ergodic Theorem, to derive for example the average number of occurrences of certain digits in typical expansions. The Birkhoff Ergodic Theorem for ergodic transformations says the following.

Theorem 1.2.2 (Birkhoff Ergodic Theorem for ergodic transformations)**.** *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system with T an ergodic transformation. Then for each function $f \in L^1(\mu)$ and for μ a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_X f d\mu.$$

We use T_r to illustrate this theorem. Take, for example, the interval $\left[\frac{i}{r}, \frac{i+1}{r}\right)$ and let $1_{[\frac{i}{r}, \frac{i+1}{r})}$ denote the indicator function for this interval. Then the Ergodic Theorem says that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[\frac{i}{r}, \frac{i+1}{r})}(T_r^k x) = \int_{[0,1]} 1_{[\frac{i}{r}, \frac{i+1}{r})} d\lambda = \frac{1}{r} \quad \text{a.e.}$$

This implies that for almost all x, the fraction of digits in its r-adic expansion that is equal to i is $1/r$.

1.2.4 *r-adic transformations and isomorphisms*

We would like to have a notion of when two dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are the same. Each dynamical system has two important structures, the first one being the measure structure, given by the σ-algebra and the measure. The second structure is given by the dynamics of the transformation.

Definition 1.2.8. Two dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *isomorphic* if there exists a map $\theta : (X, \mathcal{F}, \mu, T) \to (Y, \mathcal{C}, \nu, S)$, that satisfies all the following.

(i1) θ is almost everywhere one-to-one and onto. By this we mean that if we remove a suitable μ-null set N_X from X and a suitable ν-null set N_Y from Y, such that $T(X \backslash N_X) \subset X \backslash N_X$ and $S(Y \backslash N_Y) \subset Y \backslash N_Y$, then $\theta : X \backslash N_X \to Y \backslash N_Y$ is a bijection.

(i2) θ is measurable, i.e., $\theta^{-1}(C) \in \mathcal{F}$, for all $C \in \mathcal{C}$.

(i3) θ preserves the measures, i.e., $\nu(C) = (\mu \circ \theta^{-1})(C)$ for all $C \in \mathcal{C}$.

(i4) θ preserves the dynamics of T and S, i.e., $\theta \circ T = S \circ \theta$.

We show that the dynamical system $([0,1), \mathcal{B}, \lambda, T_r)$ is isomorphic to the uniform Bernoulli shift on the symbols $0, 1, \ldots, r-1$, which is given by $(Y, \mathcal{C}, \nu, \sigma)$ with $Y = \{0, 1, \ldots, r-1\}^\omega$, $\mathcal{C}$ the product σ-algebra on Y, ν the uniform product measure on $(Y, \mathcal{C})$ and σ the left-shift. Define the map $\theta : [0,1) \to Y$ by

$$\theta(x) = \theta\left(\sum_{k=1}^{\infty} \frac{b_k}{r^k}\right) = b_1 b_2 \cdots ,$$

where $b_1 b_2 \cdots$ is the digit sequence of x, generated by T_r. Let

$$C(d_1 \cdots d_n) = \{u \in Y \mid u_k = d_k,\ 1 \leq k \leq n\}$$

denote a cylinder set of length n. Since the cylinder sets generate $\mathcal{C}$, it is enough to check measurability and measure preservingness on the cylinders. Since

$$\theta^{-1}(C(d_1 \cdots d_n)) = \left[\frac{d_1}{r} + \cdots + \frac{d_n}{r^n}, \frac{d_1}{r} + \cdots + \frac{d_n + 1}{r^n}\right),$$

and

$$\lambda\big(\theta^{-1}(C(d_1 \cdots d_n))\big) = \frac{1}{r^n} = \nu(C(d_1 \cdots d_n)),$$

this is obviously true. Note that the set

$$N = \{u \in Y \mid \exists k \geq 1 \,:\, y_j = r - 1 \text{ for all } j \geq k\}$$

is a subset of Y of measure zero. Then $\theta : [0,1) \to Y \backslash N$ is a bijection. It is clear that $\theta \circ T = \sigma \circ \theta$. This proves (i1)-(i4).

Definition 1.2.9. Let T be a transformation that is defined on a complete probability space $(X, \mathcal{F}, \mu)$. If the system $(X, \mathcal{F}, \mu, T)$ is isomorphic to the completion of a Bernoulli shift, then T is called *Bernoulli*.

Thus the r-adic transformation T_r, when defined on the space $([0,1), \mathcal{L}, \lambda)$, is isomorphic to the one-sided Bernoulli shift and hence, it is Bernoulli.

1.2.5 *r-adic transformations and mixing properties*

For dynamical systems, there are several notions of independence. These notions indicate the degree of independence that a system has. The weakest form of independence is ergodicity. The strongest form is Bernoullicity. In between there are several degrees of mixing.

Definition 1.2.10. Let T be a measure preserving transformation on a probability space $(X, \mathcal{F}, \mu)$. Then

(i) T is called *weakly mixing* if for each $A, B \in \mathcal{F}$,

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| = 0.$$

(ii) T is called *strongly mixing* if for each $A, B \in \mathcal{F}$,

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

There is a generalized notion of mixing, that is mixing of order k.

Definition 1.2.11. Let T be a measure preserving transformation on a probability space $(X, \mathcal{F}, \mu)$. Then T is called *mixing of order k* with $k \geq 1$ if for each sequence of k-tuples of integers $t_n^1, \ldots, t_n^k$ such that $\lim_{n\to\infty} \inf_{1\leq i,j\leq k} |t_n^i - t_n^j| = \infty$ and for all sets $A_0, \ldots, A_k \in \mathcal{F}$, we have

$$\lim_{n\to\infty} \mu(T^{-t_n^k} A_k \cap \cdots \cap T^{-t_n^1} A_1 \cap A_0) = \mu(A_k) \cdots \mu(A_1)\mu(A_0).$$

These properties of dynamical systems are called *mixing properties*. We have the following proposition.

Proposition 1.2.1. *All mixing properties are preserved under isomorphism.*

The proof of this last proposition can be found in all standard textbooks on ergodic theory. See for example [Wal82] by Walters.

Another form of independence is called weakly Bernoulli. In a system that is weakly Bernoulli, the future and distant past are approximately independent. The precise definition is as follows.

Definition 1.2.12. A dynamical system $(X, \mathcal{F}, \mu, T)$ is called *weakly Bernoulli* if for each ε there is a positive integer N, such that for all $m \geq 1$ and for all $A \in \bigvee_{k=0}^{m} T^{-k}\mathcal{F}$ and $C \in \bigvee_{k=-N-m}^{-N} T^{-k}\mathcal{F}$, we have

$$|\mu(A \cap C) - \mu(A)\mu(C)| < \varepsilon,$$

where $\bigvee_{k=j}^{n} T^{-k}\mathcal{F}$, $j, n \in \mathbb{Z}$, is the smallest σ-algebra, containing all the σ-algebras $T^{-k}\mathcal{F}$.

Since T_r is Bernoulli, the transformation is also all other forms of mixing, as well as weakly Bernoulli. Another interesting property for a transformation is exactness.

Definition 1.2.13. A measure preserving transformation T, defined on a probability space $(X, \mathcal{F}, \mu)$ is called *exact* if the set $\bigcup_{k=0}^{\infty} T^{-k}\mathcal{F}$ only contains sets of μ-measure 0 or 1.

Exact transformations turn out to have very nice properties. Rohlin gave an extensive characterization of exact transformations in [Roh61]. For example, he proved that a transformation T is exact, if and only if for every set E with positive measure and with measurable images TE, T^2E, $\ldots$, it holds that $\lim_{n\to\infty} \mu(T^n E) = 1$. He also showed that exact transformations are mixing of all orders. The next theorem gives a way to check if a transformation is exact.

Theorem 1.2.3 (Rohlin, [Roh61]). *Let $(X, \mathcal{F}, \mu)$ be a Lebesgue space and $\mathcal{C}$ a countable collection of sets of positive measure, such that every set from $\mathcal{F}$ can be expressed as a countable union of elements from $\mathcal{C}$. Suppose there is a positive, integer valued function $n(C)$ for $C \in \mathcal{C}$, such that $\mu(T^{n(C)}C) = 1$. Suppose also that there is a positive integer γ, such that for all measurable sets $E \subseteq C$ with measurable image $T^{n(C)}E$ we have*

$$\mu(T^{n(C)}E) \leq \gamma \cdot \frac{\mu(E)}{\mu(C)}.$$

Then T is exact.

We show that T_r is exact. Consider the intervals

$$\Delta(d_1 \cdots d_n) = \left[\frac{d_1}{r} + \cdots + \frac{d_n}{r^n}, \frac{d_1}{r} + \cdots + \frac{d_n + 1}{r^n}\right).$$

It is clear that these intervals satisfy all the conditions of Theorem 1.2.3 with $n(\Delta(d_1 \cdots d_n)) = n$. Take such a set $\Delta(d_1 \cdots d_n)$. Then $\lambda(\Delta(d_1 \cdots d_n)) = \frac{1}{r^n}$. Set $\gamma = 1$. For all measurable subsets $E \subseteq \Delta(d_1 \cdots d_n)$ we have

$$\lambda(T_r^n E) = r^n \lambda(E) = \frac{\lambda(E)}{\lambda(\Delta(d_1 \cdots d_n))}.$$

Hence, T_r is exact.

1.2.6 *r-adic transformations and entropy*

The entropy of a dynamical system reflects the amount of randomness that is generated by the system. Let $(X, \mathcal{F}, \mu)$ be a probability space and T a measure preserving transformation. The measure theoretical entropy measures the average uncertainty about where T moves the points of X. It is defined in a few steps.

Definition 1.2.14. Let I be a finite or countable index set. A *partition* α of the measure space $(X, \mathcal{F}, \mu)$ is a finite or countable collection of measurable subsets of X, $\alpha = \{\alpha_i \mid i \in I\}$, such that the following hold.

(i) $\mu(\alpha_i \cap \alpha_j) = 0$, $i \neq j$.

(ii) $\mu(X \setminus \bigcup_{i \in I} \alpha_i) = 0$.

For a finite partition $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ of X, the *entropy of the partition* is given by

$$H(\alpha) = -\sum_{i=1}^{n} \mu(\alpha_i) \log \mu(\alpha_i).$$

Given a finite partition α of X, the object $\bigvee_{k=0}^{n-1} T^{-k}\alpha$ is the partition, whose elements are all sets of the form

$$\alpha_{i_0} \cap T^{-1}\alpha_{i_1} \cap \cdots \cap T^{-(n-1)}\alpha_{i_{n-1}}.$$

Then the entropy of the transformation T with respect to the partition α is given by

$$h(\alpha, T) = h_\mu(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} T^{-k}\alpha\right).$$

Finally, the next definition says what the measure theoretical entropy of the transformation T is.

Definition 1.2.15. Let $(X, \mathcal{F}, \mu)$ be a measure space and T a measure preserving transformation. Then the *measure theoretical entropy* of T is given by

$$h_\mu(T) = \sup_\alpha h(\alpha, T),$$

where the supremum is taken over all partitions α that have finite entropy.

For continuous transformations $T : X \to X$, defined on a compact topological space X, we can define another notion of entropy, called the topological entropy. For an open cover ζ of X, let $N(\zeta)$ denote the number of sets in a finite subcover of ζ with smallest cardinality. The entropy of the cover ζ is $H(\zeta) = \log N(\zeta)$. The entropy of the continuous map T with respect to the cover ζ is then

$$h(T, \zeta) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} T^{-k}\zeta\right),$$

where $\bigvee_{k=0}^{n-1} T^{-k}\zeta$ is the open cover by all sets of the form $A_0 \cap \cdots \cap A_{n-1}$ with $A_k \in T^{-k}\zeta$, $0 \le k \le n-1$.

Definition 1.2.16. Let X be a compact topological space and $T : X \to X$ a continuous transformation. Then the *topological entropy* of T is given by

$$h(T) = \sup_\zeta h(T, \zeta),$$

where the supremum is taken over all open covers of X.

For continuous transformations on compact metric spaces, we have the following relation between the measure theoretical entropy and the topological entropy. If X is a compact metric space, then we use $\mathcal{B}(X)$ to denote the Borel σ-algebra on X. If $T : X \to X$ is a continuous transformation, then $M(X, T)$ is the set of all T-invariant probability measures on $(X, \mathcal{B}(X))$.

Theorem 1.2.4 (Variational Principle)**.** *Let X be a compact metric space and $T : X \to X$ a continuous map. Then*

$$h(T) = \sup\{h_\mu(T) \mid \mu \in M(X, T)\}.$$

For the proof of Theorem 1.2.4 and more information on entropy in general, see [Wal82]. This theorem makes the following definition possible.

Definition 1.2.17. Let $T : X \to X$ be a continuous transformation on a compact metric space X. Then a measure $\mu \in M(X, T)$ is called a *measure of maximal entropy* for T if $h_\mu(T) = h(T)$.

For a two-sided Bernoulli shift on n symbols with product measure μ_p, given by $p = (p_1, \dots, p_n)$, the measure theoretical entropy is $-\sum_{k=1}^{n} p_k \log p_k$. The topological entropy of the two-sided Bernoulli shift on n symbols is $\log n$ and this is equal to the measure theoretical entropy of the uniform product measure. Hence, the uniform product measure is a measure of maximal entropy for the two-sided Bernoulli shift. The following theorem can be found in standard textbooks. For more information, see for example [DK02a].

Theorem 1.2.5. *Entropy is isomorphism invariant.*

By the isomorphism between the dynamical systems of T_r and of the uniform Bernoulli shift on r symbols, the measure theoretical entropy of the transformation T_r with respect to the Lebesgue measure is equal to

$$h_\lambda(T_r) = -\sum_{k=1}^{r} \frac{1}{r} \log\left(\frac{1}{r}\right) = \log r.$$

Since in Theorem 1.2.5 isomorphism is intended as in Definition 1.2.8, i.e., in a measure theoretical sense, Theorem 1.2.4 can also give an upper bound for the measure theoretical entropy of transformations that are not continuous. For example, since the topological entropy of the uniform Bernoulli shift on r symbols is equal to $\log r$, the Lebesgue measure is a measure of maximal entropy for T_r.

1.2.7 *r-adic transformations and GLS-transformations*

The transformation T_r belong to the class of GLS-transformations. GLS stands for *generalized Lüroth series*. These transformations were first introduced in [DK02a]. They are mentioned here, because they can also be used to generate number expansions and display very simple dynamics. The definition of the GLS-transformation is as follows. Let $\mathcal{D} \subset \mathbb{Z}_+$ be a finite or countable set of positive integers. A partition $\mathcal{I} = \{I_k = [\ell_k, r_k) \,|\, k \in \mathcal{D}\}$ of the interval $[0,1)$ is called a *GLS-partition* if for $L_k = r_k - \ell_k$ we have $\sum_{k \in \mathcal{D}} L_k = 1$ and $0 < L_i \leq L_j < 1$ for all $i, j \in \mathcal{D}$ with $i < j$. Set $I_\infty = [0,1) \setminus \bigcup_{k \in \mathcal{D}} I_k$. The GLS-transformation $S : [0,1) \to [0,1)$ is given by

$$Sx = \begin{cases} \dfrac{x}{r_k - \ell_k} - \dfrac{\ell_k}{r_k - \ell_k}, & \text{if } x \in I_k,\ k \in \mathcal{D}, \\[2ex] 0, & \text{if } x \in I_\infty. \end{cases}$$

Note that a GLS-transformation is completely determined by the partition $\mathcal{I}$. Hence, for a GLS-partition $\mathcal{I}$, we can speak of the GLS($\mathcal{I}$)-transformation. An arbitrary example is given in Figure 1.3.

The GLS-transformations have very nice properties, due to the fact that they map each of the intervals I_k to the complete interval $[0,1)$. For example, it is straightforward to check that they are invariant with respect to the Lebesgue measure. We can obtain number expansions from a GLS-transformation in the following way. We first need some notation. For $x \in I_k$,

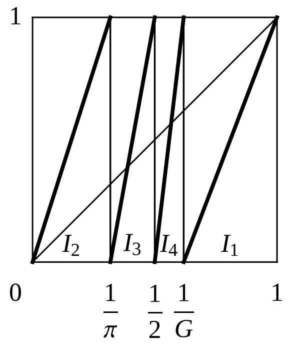


Figure 1.3 The GLS-transformation with $\mathcal{D} = \{1, 2, 3, 4\}$ and $I_1 = [1/G, 1)$, $I_2 = [0, 1/\pi)$, $I_3 = [1/\pi, 1/2)$ and $I_4 = [1/2, 1/G)$, where $G = \frac{1+\sqrt{5}}{2}$ is the golden mean.

set

$$s(x) = \frac{1}{r_k - \ell_k} \quad \text{and} \quad d(x) = \frac{\ell_k}{r_k - \ell_k},$$

so that $Sx = xs(x) - d(x)$. So, in a sense $s(x)$ is the local derivative of S and $d(x)$ corresponds to the digit. Now, for $n \geq 1$ let

$$s_n = s_n(x) = \begin{cases} s(S^{n-1}x), & \text{if } S^{n-1}x \in \bigcup_{k\in\mathcal{D}} I_k, \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$d_n = d_n(x) = \begin{cases} d(S^{n-1}x), & \text{if } S^{n-1}x \in \bigcup_{k\in\mathcal{D}} I_k, \\ 1, & \text{otherwise.} \end{cases}$$

So, if $S^n x \in \bigcup_{k\in\mathcal{D}} I_k$, $n \geq 0$, then we can write

$$\begin{aligned} x &= \frac{d_1}{s_1} + \frac{Sx}{s_1} = \frac{d_1}{s_1} + \frac{d_2}{s_1 s_2} + \frac{S^2 x}{s_1 s_2} \\ &= \frac{d_1}{s_1} + \frac{d_2}{s_1 s_2} + \cdots + \frac{d_n}{s_1 s_2 \cdots s_n} + \frac{S^n x}{s_1 s_2 \cdots s_n}. \end{aligned}$$

Since $S^n x \in [0, 1)$ for all $n \geq 0$ and since $s_j > 1$ for all $j \geq 1$, this series converges and gives the GLS($\mathcal{I}$)-expansion of x. This expansion can be represented by the sequence $(d_n, s_n)_{n\geq 1}$. To know all the values of d_n and s_n, we only need to know in which of the partition elements x lies after each iteration. We can define the digit sequence $(b_n)_{n\geq 1}$ of x under S by setting $b_n = k$ if $S^{n-1}x \in I_k$. As in the r-adic case, the transformation S acts like the left-shift on these sequences. Notice that S can generate all sequences from $\mathcal{D}^\omega$. This implies that the transformation S is Bernoulli. For more information on GLS-transformations, see [DK02a]. We will encounter them again in Section 3.1.

The r-adic transformation T_r on $[0, 1)$ is a GLS-transformation with $\mathcal{D} = \{1, \ldots, r\}$ and $I_k = \left[\frac{k-1}{r}, \frac{k}{r}\right)$ for all $k \in \mathcal{D}$. Then each interval has length $\frac{1}{r}$.

So, if $x \in I_k$, then $s(x) = r$ and $d(x) = \frac{k-1}{r} \cdot r = k - 1$. Hence, on I_k we have $Sx = rx - (k - 1)$.

1.3 In this dissertation

In this dissertation, we will address all the subjects mentioned above, with respect to β-expansions with arbitrary digits.

In Chapter 2 we define a transformation that generates the greedy β-expansions with arbitrary digits. This chapter is mainly concerned with finding the invariant measure for the transformation. We prove the existence of a unique invariant measure, absolutely continuous with respect to the Lebesgue measure and show that it is ergodic. We also identify the support of this measure.

In Chapter 3 we focus on digit sets that have only three digits and construct another very useful object from ergodic theory, the natural extension. This natural extension is a way to make the transformation invertible. In general the dynamics of the transformation is not invertible. One point has many possible inverse images. This implies that after an application of the transformation, there is no way to remember the 'past'. Each time the transformation is applied, information is lost. When a transformation is invertible, this is not the case. One can see the future as well as the past. Aside from this, the measure theoretical natural extension has many other useful properties. For one, it allows us to obtain an explicit expression for the invariant measure of the transformation. Chapter 3 also deals with this.

Chapter 4 is about expansions in general. For non-integer bases, almost all points that have a β-expansion, have infinitely many different expansions. So, there is no longer just one way to obtain expansions dynamically. We introduce the opposite of the greedy transformation, which is called the lazy transformation, and after that we give a whole family of transformations that can generate β-expansions with arbitrary digits. We give a random transformation that, for a given base and a given digit set, generates all the possible expansions of an x in that base and with that digit set. In the last section of Chapter 4, we review some known results on the size of the sets of numbers that have an expansion in a certain base and with certain digits.

The last chapter, Chapter 5, is concerned with expansions of which the base has special algebraic properties. The base β will be a Pisot unit. This allows us to represent the expansions in a nice geometrical way. This representation gives further information about the expansions. The main theorem of this chapter says that the representations give multiple tilings of certain Euclidean spaces.

CHAPTER 2

THE GREEDY β-TRANSFORMATION

2.1 Know your classics

The classical greedy β-expansions are a first natural generalization of the r-adic expansions. Instead of an integer, they can have any real number bigger than 1 as a base. Although the step from integers to non-integers seems small, results are far more difficult to obtain and differ enormously from the r-adic case. We sketch some of these results below.

Let $\beta > 1$ be a non-integer and consider the set of digits $A_\beta = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. Recall that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Expressions of the form

$$x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}, \tag{2.1}$$

where $b_k \in A_\beta$ for all $k \geq 1$, are called *classical β-expansions*. The word 'classical' refers to the digit set. One way to obtain such expansions, is by the recursive *greedy algorithm*. Suppose that the digits $b_1, \ldots, b_{n-1}$ are already known. Then the n-th digit b_n can be chosen as the largest element from A_β, such that

$$\frac{b_1}{\beta} + \cdots + \frac{b_n}{\beta^n} \leq x.$$

This is called the greedy algorithm, since at each step it chooses the largest digit possible. The expansions with digits in A_β that are produced by the greedy algorithm are called the *classical greedy β-expansions*. The smallest number that can be obtained in this manner, is by using 0's for all b_k's and that gives us $x = 0$. Similarly, the largest number we can get is $x = \frac{\lfloor \beta \rfloor}{\beta - 1}$. So, if a number has an expansion of the form (2.1) with digits in A_β, then it lies in the interval $\left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$.

A first important difference between β-expansions and r-adic expansions is the number of different expansions that an element can have. In the Introduction we saw that for each integer $r > 1$, almost all numbers in the interval $[0,1]$ have a unique r-adic expansion and all numbers that do not have a unique expansion, have exactly two. If $\beta > 1$ is a non-integer, then almost all $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ have a continuum of expansions in base β and digits in A_β, see [EJK90], [Sid03], [DdV07].

As in the r-adic case, expansions from (2.1) can be generated using the iterations of a transformation. For classical β-expansions, one such transformation is T_β, which is defined from the interval $\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ to itself by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & \text{if } 0 \le x < 1, \\ \\ \beta x - \lfloor\beta\rfloor, & \text{if } 1 \le x \le \frac{\lfloor\beta\rfloor}{\beta-1}. \end{cases}$$

There exist other transformations that generate classical β-expansions. We will discuss them in Chapter 4. In Figure 2.1 we see T_π.

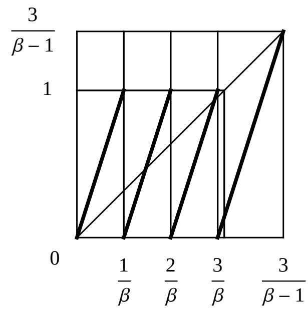


Figure 2.1 The greedy β-transformation for $\beta = \pi$. Then $A_\beta = \{0,1,2,3\}$.

Using this transformation, we can make for each $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ a digit sequence $b(x) = (b_{\beta,k}(x))_{k\ge1}$. Set

$$b_1 = b_{\beta,1}(x) = \begin{cases} i, & \text{if } x \in \left[\frac{i}{\beta}, \frac{i+1}{\beta}\right), \text{ for } i \in \{0,\dots,\lfloor\beta\rfloor - 1\}, \\ \\ \lfloor\beta\rfloor, & \text{if } x \in \left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right], \end{cases}$$

and for $n \ge 1$, set $b_n = b_{\beta,n}(x) = b_1(T_\beta^{n-1}x)$. Then $T_\beta x = \beta x - b_1$ and inverting this relation gives $x = \frac{b_1}{\beta} + \frac{T_\beta x}{\beta}$. Thus, for any $n \ge 1$,

$$x = \sum_{k=1}^{n} \frac{b_k}{\beta^k} + \frac{T_\beta^n x}{\beta^n}.$$

Letting $n \to \infty$, it is easily seen that $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$. The transformation T_β is defined in such a way that the expansions it generates are exactly the expansions given by the greedy algorithm. Therefore T_β is called the *classical greedy β-transformation*. If $b(x) = (b_k(x))_{k\ge1}$, then $b(Tx) = (b_{k+1}(x))_{k\ge1}$. So, as in the r-adic case, the transformation T_β behaves like the left-shift on the digit sequences.

T_β has been the object of extensive studies from many points of view. From an ergodic viewpoint, the existence of an invariant measure, absolutely continuous with respect to the Lebesgue measure, is an important feature of the transformation. From Chapter 1 we know that for each integer $r > 1$, the Lebesgue measure λ is an invariant measure for the transformation $T_r : x \mapsto rx \pmod 1$, defined on the unit interval. For the transformation T_β, this does not hold, but T_β does have a unique invariant measure, absolutely continuous with respect to λ. Rényi proved the existence of this measure ([Rén57]) and Gel'fond and Parry independently gave an explicit formula for the density function of this measure in [Gel59] and [Par60] respectively. The invariant measure has the unit interval $[0, 1)$ as its support and the density function h_β is given by

$$h_\beta : [0,1) \to [0,1) : x \mapsto \frac{1}{F(\beta)} \sum_{k=0}^{\infty} \frac{1}{\beta^k} 1_{[0,T_\beta^k 1)}(x), \tag{2.2}$$

where $F(\beta) = \int_0^1 \sum_{x<T_\beta^k 1} \frac{1}{\beta^k} d\lambda$ is a normalizing constant. Notice the importance of the point 1 in this formula. At a first glance, the formulas from both Kopf ([Kop90]) and Góra ([Gór09]), mentioned in Section 1.2.3, look like the density from (2.2). This is not surprising, since the greedy β-transformation is contained in the class of maps studied by both. In general, however, the entries of their matrices M and S are not so easy to obtain. This makes it difficult to use their techniques to get a formula for the density similar to (2.2).

As we have seen in Section 1.2.2, for an integer $r > 1$, the transformation T_r produces all elements from $\{0, \dots, r-1\}^\omega$ as digit sequences, except for the ones ending in an infinite string of $(r-1)$'s. For the classical greedy β-transformation, this is no longer true. There are sequences in A_β^ω that do not occur as digit sequences of T_β. In [Par64] Parry gave a complete characterization of the digit sequences that are generated by the transformation T_β. Here, again, the expansion of 1 plays an important role. Recall that $\prec$ denotes the lexicographical ordering. Let $(d_k)_{k\geq 1}$ denote the digit sequence of 1 under T_β, i.e., $d_k = b_k(1)$ for all $k \geq 1$. Now, define the sequence $(d_k^*)_{k\geq 1}$ as follows. If $(d_k)_{k\geq 1}$ does not end in an infinite string of 0's, let $(d_k^*)_{k\geq 1}$ be equal to $(d_k)_{k\geq 1}$. If $(d_k)_{k\geq 1}$ does end in an infinite string of 0's, then we can write it as $d_1 \cdots d_n 0^\omega$ with $d_n \neq 0$ for some $n > 1$. In this case, let $(d_k^*)_{k\geq 1}$ be the sequence $(d_1 \cdots d_{n-1}(d_n - 1))^\omega$. Let $x \in [0, 1)$ and suppose that $x = \sum_{k=1}^\infty \frac{b_k}{\beta^k}$ is an expansion of x with each $b_k \in A_\beta$. Parry proved the following theorem.

Theorem 2.1.1 (Parry, [Par64]). *The expansion $x = \sum_{k=1}^\infty \frac{b_k}{\beta^k}$ is the expansion of x generated by T_β if and only if for each $n \geq 1$,*

$$b_n b_{n+1} \cdots \prec d_1^* d_2^* \cdots .$$

So, the step from integer bases to non-integer bases is a very non-trivial one. We will go one step further and generalize the digit set from A_β to arbitrary sets of real numbers. In 2005, Pedicini ([Ped05]) gave an algorithm to produces such β-expansions with arbitrary digits. He proved, among other things, that under a certain condition on the distance between two consecutive elements of the digit set, every number in some interval has a β-expansion with digits in this set. Pedicini also gave a characterization of the set of sequences that his algorithm produces. We will state some of his results in a precise manner in the next section. There, we will also introduce a transformation that generates β-expansions with digits in an arbitrary set of real numbers and prove some properties of this transformation. In Section 2.3 we will establish the existence of a unique invariant measure, absolutely continuous with respect to the Lebesgue measure, for this transformation. Sections 2.4 and 2.5 are devoted to two situations in which we can give an explicit expression for the invariant measure. The transformations that will be defined in Section 2.2 are isomorphic to transformations that fall into the class of maps studied by Kopf ([Kop90]) and Góra ([Gór09]). Therefore, we already have expressions for the invariant measures of such transformations. However, as mentioned before, the matrices that are used to give these expressions, are somewhat difficult to obtain. In Section 2.4, Section 2.5 and Chapter 3 we will use other techniques to obtain a formula for the invariant measure of the transformations under consideration in some special cases. The expression obtained in this way is clearly a generalization of formula (2.2). In Section 2.4 we put a restriction on the number of elements in the digit set. In Section 2.5 we give conditions under which the dynamics of the transformation can be described by a Markov chain. In Chapter 3 we will say more about invariant measures for β-transformations with three digits using a natural extension of the dynamical system.

2.2 It works to be greedy

In 2005 ([Ped05]), Pedicini defined an algorithm that generates β-expansions with arbitrary digits. The next definition states what is meant by β-expansions with digits in some set.

Definition 2.2.1. Let $\beta > 1$ be a real number and let $A = \{a_0, \ldots, a_m\}$ be a set of $m + 1$ arbitrary real numbers. An expression of the form

$$x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}, \tag{2.3}$$

such that $b_k \in A$ for all $k \geq 1$ is called a *β-expansion with digits in A* or, if the set A is not specified, a *β-expansion with arbitrary digits*. If the sequence $(b_k)_{k\geq 1}$ is generated by a transformation T, then the sequence $b(x) = (b_k)_{k\geq 1}$ is called the *digit sequence of x, generated by T*. We usually write $b_1 b_2 \cdots$ instead of $(b_k)_{k\geq 1}$. Sometimes we also write $x = .b_1 b_2 \cdots$, which is intended as $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^i}$. Of course, this notation depends on the value of β. We will only use it, when there is no possibility for confusion.

Note that the smallest number that can be represented as in (2.3) is obtained if we take $b_k = a_0$ for all $k \geq 1$. This gives $\frac{a_0}{\beta-1}$. Similarly, the largest number we can get is $\frac{a_m}{\beta-1}$. Hence, all numbers of the form (2.3) are elements of the interval $\left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$. Pedicini's algorithm gives expressions of the form (2.3) for some set A. His algorithm is similar to the algorithm that gives the classical greedy expansions and is defined recursively as follows. Let $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$, and suppose the digits $b_1 = b_1(x), \dots, b_{n-1} = b_{n-1}(x)$ are already known; then $b_n = b_n(x)$ is the largest element of $\{a_0, \dots, a_m\}$, such that

$$\frac{b_1}{\beta} + \cdots + \frac{b_n}{\beta^n} + \sum_{k=n+1}^{\infty} \frac{a_0}{\beta^k} \leq x. \tag{2.4}$$

Pedicini showed that if the digit set A satisfies the following condition,

$$\max_{0 \leq i \leq m-1} (a_{i+1} - a_i) \leq \frac{a_m - a_0}{\beta - 1}, \tag{2.5}$$

then every point in $\left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a β-expansion with digits in A and satisfying (2.4). In fact, Sidorov proved ([Sid07]) that if strict inequality holds in condition (2.5), then a.e. $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a continuum of β-expansions of the form (2.3). Furthermore, for a fixed digit set A there exists a $\beta_0 > 1$ such that for each $\beta \in (1, \beta_0)$, every x in this interval has a continuum of expansions (2.3).

Using this greedy algorithm we can define a transformation, that generates these expansions by iteration, just like in the classical case. We want such a transformation to behave like the left-shift on the digit sequences, i.e., we are seeking a transformation T on $\left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ such that if (2.3) is the greedy expansion of x as given by (2.4), then $T^n x = \sum_{k=1}^{\infty} \frac{b_{n+k}}{\beta^k}$. Assume that the digit set is ordered, i.e., that $a_0 < a_1 < \cdots < a_m$. Suppose that we already know the expansion of x given by the greedy algorithm, $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$. Then, if we rewrite condition (2.4), we see that for $i \in \{0, \dots, m-1\}$,

$$\begin{aligned}
b_n = a_i \quad &\Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{a_i}{\beta^n} + \sum_{k=n+1}^{\infty} \frac{a_0}{\beta^k} \leq x < \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{a_{i+1}}{\beta^n} + \sum_{k=n+1}^{\infty} \frac{a_0}{\beta^k} \\
&\Leftrightarrow \quad \frac{a_0}{\beta^n(\beta-1)} + \frac{a_i}{\beta^n} \leq \frac{1}{\beta^{n-1}} \sum_{k=1}^{\infty} \frac{b_{n-1+k}}{\beta^k} < \frac{a_0}{\beta^n(\beta-1)} + \frac{a_{i+1}}{\beta^n} \\
&\Leftrightarrow \quad \frac{a_0}{\beta-1} + \frac{a_i - a_0}{\beta} \leq \sum_{k=1}^{\infty} \frac{b_{n-1+k}}{\beta^k} < \frac{a_0}{\beta-1} + \frac{a_{i+1} - a_0}{\beta},
\end{aligned}$$

and $b_n = a_m$ if and only if

$$\frac{a_0}{\beta-1} + \frac{a_m - a_0}{\beta} \leq \sum_{k=1}^{\infty} \frac{b_{n-1+k}}{\beta^k} \leq \frac{a_m}{\beta-1}.$$

In view of this, we define the greedy transformation $T = T_{\beta,A}$ with digit set A satisfying (2.5) by

$$Tx = \begin{cases} \beta x - a_i, & \text{if } x \in \left[\dfrac{a_0}{\beta-1} + \dfrac{a_i - a_0}{\beta}, \dfrac{a_0}{\beta-1} + \dfrac{a_{i+1} - a_0}{\beta}\right), \\ & \text{for } i \in \{0, \dots, m-1\}, \\ \\ \beta x - a_m, & \text{if } x \in \left[\dfrac{a_0}{\beta-1} + \dfrac{a_m - a_0}{\beta}, \dfrac{a_m}{\beta-1}\right]. \end{cases}$$

The first proposition will allow simplifications to the definition of T.

Proposition 2.2.1. *Let T be the greedy β-transformation for a certain real $\beta > 1$ and a digit set $A = \{a_0, a_1, \dots, a_m\}$ satisfying (2.5). Let $A_0 = \{0, a_1 - a_0, \dots, a_m - a_0\}$ be the digit set, obtained from A by subtracting the first digit from all the digits in the set. Let T_0 be the corresponding greedy β-transformation with digits in A_0. Define the function $\theta : \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right] \to \left[0, \frac{a_m - a_0}{\beta-1}\right]$ by*

$$\theta(x) = x - \frac{a_0}{\beta - 1}.$$

Then θ is a continuous bijection and $T_0 \circ \theta = \theta \circ T$.

Proof. It is obvious that θ is a continuous bijection. To show that $T_0 \circ \theta = \theta \circ T$, let $x \in \left[\frac{a_0}{\beta-1} + \frac{a_i - a_0}{\beta}, \frac{a_0}{\beta-1} + \frac{a_{i+1} - a_0}{\beta}\right)$ for some $i \in \{0, 1, \dots, m-1\}$. Then

$$\theta(Tx) = \beta x - a_i - \frac{a_0}{\beta - 1}.$$

On the other hand, $\theta(x) \in \left[\frac{a_i - a_0}{\beta}, \frac{a_{i+1} - a_0}{\beta}\right)$, so

$$(T_0 \circ \theta)(x) = \beta\left(x - \frac{a_0}{\beta - 1}\right) - (a_i - a_0) = \beta x - a_i - \frac{\beta a_0}{\beta - 1} + a_0 = \theta(Tx).$$

A similar proof can be given for $x \in \left[\frac{a_0}{\beta-1} + \frac{a_m - a_0}{\beta}, \frac{a_m}{\beta-1}\right]$. □

This lemma implies that every greedy transformation with digit set A is isomorphic to a greedy transformation with a digit set of which the first digit is equal to zero. So, w.l.o.g., we can assume that every digit set A has zero as the first digit.

Definition 2.2.2. Let $\beta > 1$ be a real number. A set of real numbers $A = \{a_0, a_1, \dots, a_m\}$ is called an *allowable digit set* for β if it satisfies all three conditions below.

(i) $a_0 < a_1 < \cdots < a_m$,

(ii) $a_0 = 0$,

(iii) $\displaystyle\max_{0 \le i \le m-1} (a_{i+1} - a_i) \le \frac{a_m}{\beta - 1}$.

The definition of the greedy transformation now becomes the following.

Definition 2.2.3. Let $\beta > 1$ be a real number and $A = \{a_0, \dots, a_m\}$ an allowable digit set for β. Then the *greedy β-transformation with digits in A*, $T = T_{\beta,A}$, is defined from the interval $\left[0, \frac{a_m}{\beta-1}\right]$ to itself by

$$Tx = \begin{cases} \beta x - a_i, & \text{if } x \in \left[\dfrac{a_i}{\beta}, \dfrac{a_{i+1}}{\beta}\right), \text{ for } i \in \{0, \dots, m-1\}, \\[2ex] \beta x - a_m, & \text{if } x \in \left[\dfrac{a_m}{\beta}, \dfrac{a_m}{\beta - 1}\right]. \end{cases}$$

In Figure 2.2 we see an example of such a transformation.

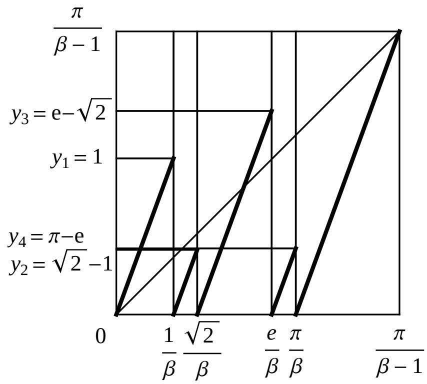


Figure 2.2 The greedy β-transformation for $\beta = 1 + \sqrt{3}$ and $A = \{0, 1, \sqrt{2}, e, \pi\}$.

Notice that $T\left[0, \frac{a_m}{\beta-1}\right] = \left[0, \frac{a_m}{\beta-1}\right]$. So, T is always surjective. Furthermore, the assumption that A is an allowable digit set implies that for $i \in \{0, 1, \dots, m-1\}$,

$$T\left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\right) = [0, a_{i+1} - a_i) \subseteq \left[0, \frac{a_m}{\beta - 1}\right].$$

This shows that T maps the interval $\left[0, \frac{a_m}{\beta-1}\right]$ onto itself. Let

$$b_1 = b_1(x) = \begin{cases} a_i, & \text{if } x \in \left[\dfrac{a_i}{\beta}, \dfrac{a_{i+1}}{\beta}\right), \text{ for } i \in \{0, \dots, m-1\}, \\[2ex] a_m, & \text{if } x \in \left[\dfrac{a_m}{\beta}, \dfrac{a_m}{\beta - 1}\right], \end{cases}$$

and set $b_n = b_n(x) = b_1(T^{n-1}x)$. Then $Tx = \beta x - b_1$, and for any $n \geq 1$,

$$x = \sum_{k=1}^{n} \frac{b_k}{\beta^k} + \frac{T^n x}{\beta^n}. \tag{2.6}$$

Letting $n \to \infty$, it is easily seen that $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$, with b_k satisfying (2.4). From the definition of the greedy map T, it is easy to see that the point $\frac{a_m}{\beta-1}$ is the only point whose greedy expansion eventually ends in the sequence $a_m a_m \cdots$.

The next propositions give some properties of the transformation T. First we show that condition (2.5) puts a restriction on the number of elements in the allowable set A. Let $\lceil x \rceil$ denote the smallest integer bigger than or equal to x.

Proposition 2.2.2. *Let $\beta > 1$ be a real number, and let $A = \{a_0 = 0, a_1, \ldots, a_m\}$ be an allowable digit set. Then $m \geq \lceil \beta \rceil - 1$.*

Proof. By assumption, $a_{i+1} - a_i \leq \frac{a_m}{\beta - 1}$ for all $0 \leq i \leq m-1$. Summing over i gives

$$a_m = \sum_{i=0}^{m-1} (a_{i+1} - a_i) \leq m \cdot \frac{a_m}{\beta - 1},$$

so $\beta - 1 \leq m$. Since m is an integer, one has that $m \geq \lceil \beta \rceil - 1$. □

The next proposition says that the usual order on $\mathbb{R}$ respects the lexicographical ordering $\prec$ on the set of digit sequences.

Proposition 2.2.3. *Let A be an allowable digit set, and suppose $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$ and $y = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}$ are the greedy expansions of x and y in base β and digits in A. Then*

$$x < y \Leftrightarrow b_1 b_2 \cdots \prec d_1 d_2 \cdots .$$

Proof. We have $b_1 b_2 \cdots \prec d_1 d_2 \cdots$ if and only if for the smallest index n such that $b_n \neq d_n$ we have $b_n < d_n$. This holds if and only if $T^{n-1}x < T^{n-1}y$ and thus if and only if

$$x = \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{T^{n-1}x}{\beta^{n-1}} < \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{T^{n-1}y}{\beta^{n-1}} = y.$$

This gives the proposition. □

More results on the lexicographical ordering and the characterization of digit sequences generated by certain transformations are given in Chapter 5. These results are stated in a more general setting, which includes the greedy transformation.

2.3 The support act

The greedy β-transformation with arbitrary digits is a specific example of a piecewise linear, expanding map with constant slope. For this class of transformations there exists a vast literature on invariant measures. In this section we establish some properties of an invariant measure for the transformation T, that is absolutely continuous with respect to the Lebesgue measure. Such a measure is called an *ACIM*. The existence of such a measure for T follows immediately from results by Lasota and Yorke ([LY73]). We will give an expression for the support of such a measure and then we will show that this measure is unique and ergodic. Here, results from [LY78] by Li and Yorke are at the basis.

Let $\beta > 1$ be a real number and let $A = \{a_0, \dots, a_m\}$ be an allowable digit set. Let T be the greedy β-transformation with digit set A. Then T is a piecewise linear, strictly increasing transformation, which has its discontinuities in the points $\frac{a_i}{\beta}$ for $i \in \{1, \dots, m\}$. Let J denote the set containing these points. Then J is finite and for each $x \in \left[0, \frac{a_m}{\beta-1}\right] \setminus J$ we have $T'x = \beta > 1$. The points in J give a partition $\Delta = \{\Delta(i)\}_{i=0}^m$ of the interval $\left[0, \frac{a_m}{\beta-1}\right]$ by setting $\Delta(m) = \left[\frac{a_m}{\beta}, \frac{a_m}{\beta-1}\right]$ and for $i \in \{0, \dots, m-1\}$, $\Delta(i) = \left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\right)$. Define for $i \in \{1, \dots, m\}$ the values y_i to be the values obtained from T by taking the limit from the left to the points $\frac{a_i}{\beta}$, i.e., $y_i = a_i - a_{i-1}$ (See Figure 2.2).

First we define different notions of invariance under the transformation T and we state the results from Lasota and Yorke in [LY73] and from Li and Yorke in [LY78]. Let μ be a Borel measure on $\left[0, \frac{a_m}{\beta-1}\right]$. A μ-integrable function h is called *an invariant function under T* if for all measurable sets E, $\int_E h d\mu = \int_{T^{-1}E} h d\mu$. We call a Lebesgue measurable set E *forward invariant under T* if $TE = E$ modulo sets of Lebesgue measure zero. It was shown in [LY73] that there exists an invariant measure, absolutely continuous with respect to the Lebesgue measure, λ, for transformations τ that are piecewise continuous with a finite set of points of discontinuity and that have a derivative bigger than 1 for points outside this finite set. In [LY78], Li and Yorke studied these invariant measures in more detail. Their results translate in the following way to our particular greedy transformation T. For T, there exist sets $B_1, \dots, B_n$ and functions $h_1, \dots, h_n$, where $n \leq m$, such that all the following hold.

(c1) For each $k \in \{1, \dots n\}$, B_k is a finite union of closed subintervals of $\left[0, \frac{a_m}{\beta-1}\right]$. Each B_k contains at least one of the elements of J in its interior. Moreover, each B_k is forward invariant under T.

(c2) If $j \neq k$, then $B_j \cap B_k$ contains at most a finite number of points.

(c3) For λ a.e. $x \in \left[0, \frac{a_m}{\beta-1}\right] \setminus J$, there is an $k \in \{1, \dots, n\}$ such that the closure of the forward orbit of x under T equals the set B_k, i.e.,

$$\Lambda(x) := \bigcap_{N=1}^{\infty} \overline{\{T^n x\}_{n=N}^{\infty}} = B_k.$$

(c4) For each $k \in \{1, \dots n\}$, B_k is the support of the function h_k, i.e., $h_k > 0$ λ a.e. on B_k and $h_k = 0$ on B_k^c. Moreover, $\int_{B_k} h_k d\lambda = 1$.

(c5) For each $k \in \{1, \dots n\}$, h_i is invariant under T and if g is invariant under T and satisfies (c4) for some k, then $g = h_k$ λ a.e.

(c6) Each function h that is invariant under T can be written as $h = \sum_{k=1}^n \gamma_k h_k$ with a suitably chosen set of constants $\{\gamma_k\}_{k=1}^n$.

(c7) If h is an invariant function and E is a measurable set, such that TE is measurable and $TE \subseteq E$ λ a.e., then $h \cdot 1_E$ is an invariant function, where 1_E denotes the indicator function of the set E.

Remark 2.3.1. The last result was proven in [LY78] for sets E such that $TE = E$ λ a.e., however the proof of Li and Yorke still holds under the weaker assumption that $TE \subseteq E$ λ a.e.

We will first make some observations about the sets B_i.

Lemma 2.3.1. *Let $I \subseteq \left[0, \frac{a_m}{\beta-1}\right]$ be a closed interval.*

(i) If I is forward invariant under T and contains at least one element of J in its interior, then $0 \in I$.

(ii) If I does not contain an element of J in its interior, then I is not forward invariant under T.

Proof. The first part of the lemma follows immediately from the fact that for each $i \in \{1, \dots, m\}$, $T\left(\frac{a_i}{\beta}\right) = 0$. For the second part it is enough to notice that if I does not contain an element of J in its interior, then $\lambda(TI) = \beta\lambda(I)$. $\square$

Remark 2.3.2. (i) As an immediate consequence of this lemma, we have that there cannot exist two or more sets B_k satisfying (c1) and (c2). To see this, suppose that the sets B_k and B_j both satisfy (c1). Then they are both forward invariant under T, so by the previous lemma there should exist numbers $0 < x_k, x_j \leq \frac{a_m}{\beta-1}$ such that $[0, x_k] \subseteq B_k$ and $[0, x_j] \subseteq B_j$, but this contradicts (c2). So by the previously stated results from [LY78], there exists a number $0 < x \leq \frac{a_m}{\beta-1}$ and a finite number of closed intervals $I_1, \dots, I_k \subseteq \left[0, \frac{a_m}{\beta-1}\right]$ with TI_j containing a closed interval of positive Lebesgue measure, such that the set

$$B := [0, x] \cup \bigcup_{j=1}^{k} I_j \tag{2.7}$$

satisfies (c1) to (c7) for an invariant probability density function h. This implies that there exists a unique invariant measure for T that is equivalent to the Lebesgue measure on B. Notice that, w.l.o.g., we can assume that B is the finite union of disjoint closed intervals. The fact that B is forward invariant implies that $T[0, x] \subseteq [0, x]$.
(ii) In [LY82] Lasota and Yorke also prove the existence of a unique measure, absolutely continuous with respect to the Lebesgue measure for a certain class of piecewise continuous maps. The transformations they consider are defined from the unit interval $[0, 1]$ to itself, they are piecewise continuous and convex and have $T'0 > 1$. Moreover, they map the left endpoints of the intervals on which they are continuous to 0 and have positive derivatives in these points. Lasota and Yorke make extensive use of the Perron-Frobenius equation (1.2) to obtain their results and also show that the dynamical systems they consider are exact. Hence, from their results we know that the greedy β-transformation with arbitrary digits is exact. Their methods give no information on the support of the invariant measure.

The next lemma states that if B contains a closed interval whose image under T is contained in itself, then B is exactly this interval.

Lemma 2.3.2. *Let h be the density function of an invariant absolutely continuous probability measure as in Remark 2.3.2 and let B be its support. Suppose that $[\alpha_1, \alpha_2] \subseteq B$ is a closed interval. If $T[\alpha_1, \alpha_2] \subseteq [\alpha_1, \alpha_2]$ λ a.e., then $[\alpha_1, \alpha_2] = B$. Consequently, $[\alpha_1, \alpha_2]$ is a forward invariant set.*

Proof. Consider the function $g = h \cdot 1_{[\alpha_1,\alpha_2]}$. Since h is an invariant function and $[\alpha_1, \alpha_2]$ satisfies $T[\alpha_1, \alpha_2] \subseteq [\alpha_1, \alpha_2]$ λ a.e., by (c7) we know that also the function g is invariant, with its support contained in the support of h. By (c6) there exists a constant c, such that $g = c \cdot h$. Now define the function

$$f = \frac{g}{\int g d\lambda}.$$

Then f is an invariant probability density function and $f = c' \cdot h$, with $c' = c/\int g d\lambda$. This means that $f = h$ λ a.e., so that $1_{[\alpha_1,\alpha_2]}(x) = 1$ for λ almost all $x \in B$. Since B is a finite union of closed intervals, it follows that $B = [\alpha_1, \alpha_2]$. By (c1), $[\alpha_1, \alpha_2]$ is forward invariant. □

Remark 2.3.3. By the same reasoning as in the proof of the previous lemma, it can be shown that T is ergodic with respect to the ACIM. To see this, let μ be the measure given by $\mu(E) = \int_E h d\lambda$ for each measurable set E and suppose that A is a measurable set such that $T^{-1}A = A$ λ a.e. and $\mu(A) > 0$. Then $TA \subseteq A$ λ a.e., so by (c7) the function $g = h \cdot 1_A$ is invariant. Following the idea of the proof of Lemma 2.3.2 gives that $1_A = 1$ λ a.e., so $\mu(A) = 1$.

By Remark 2.3.2, there exists an element $x \in \left[0, \frac{a_m}{\beta-1}\right]$ such that the support B of h contains the interval $[0, x]$ with $T[0, x] \subseteq [0, x]$. By Lemma 2.3.2, we see that $B = [0, x] = T[0, x]$ λ a.e. The next two lemmas specify the value of x. First we define the following value. Remember that for $i \in \{1, \ldots, m\}$, $y_i = a_i - a_{i-1}$. Let $y_{i_0} = \max\left\{y_i \mid \frac{a_i}{\beta} \leq x\right\}$ and if there are two or more indices for which this holds, then let y_{i_0} be the one with the smallest index.

Lemma 2.3.3. *Let $B = [0, x]$ be the support of the probability density function h as described above. Then $B = [0, y_{i_0}]$.*

Proof. Since $T[0, x] = [0, x]$ λ a.e., we have that $y_i \leq x$ for any i such that $\frac{a_i}{\beta} \leq x$. Hence $y_{i_0} \leq x$. Also $Tx \leq x$.

Suppose $x \in \Delta(k)$ for some $k \in \{0, \ldots, m\}$. Then by the definition of y_{i_0},

$$T\left[0, \frac{a_k}{\beta}\right) \subseteq [0, y_{i_0}] \quad \lambda \text{ a.e.} \tag{2.8}$$

If $y_{i_0} \in \left[0, \frac{a_k}{\beta}\right)$, then $T[0, y_{i_0}] \subseteq [0, y_{i_0}] \subseteq [0, x]$ λ a.e. and thus by Lemma 2.3.2, $[0, y_{i_0}] = [0, x]$ λ a.e. If, on the other hand, $y_{i_0} \in \left[\frac{a_k}{\beta}, x\right]$, then since $Tx \leq x$, we also have $Ty_{i_0} \leq y_{i_0}$ and this means that

$$T\left[\frac{a_k}{\beta}, y_{i_0}\right] \subseteq [0, y_{i_0}] \quad \lambda \text{ a.e.}$$

Combining this with equation (2.8) gives that $T[0, y_{i_0}] \subseteq [0, y_{i_0}] \subseteq [0, x]$ λ a.e., so again by Lemma 2.3.2 we have that $[0, y_{i_0}] = [0, x]$. □

From the previous lemma we know that x is one of the values y_i, $i \in \{1, \ldots, m\}$. The next lemma states explicitly which of these values it is.

Lemma 2.3.4. *Let y_{i_0} be defined as above. Then*

$$i_0 = \min\{i \mid T[0, y_i] \subseteq [0, y_i] \ \lambda \text{ a.e.}\}. \tag{2.9}$$

Proof. Since $[0, x] = [0, y_{i_0}]$ is the support of the invariant probability density function h, we must have by Lemma 2.3.2 that $T[0, y_i] \not\subseteq [0, y_i]$ λ a.e. for any $y_i < y_{i_0}$. In particular, by the definition of y_{i_0} we have that if $i < i_0$, then

$$\frac{a_i}{\beta} < \frac{a_{i_0}}{\beta} \leq y_{i_0}.$$

This implies that $y_i < y_{i_0}$ and thus that $T[0, y_i] \not\subseteq [0, y_i]$ λ a.e. Hence $i_0 = \min\{i \mid T[0, y_i] \subseteq [0, y_i] \ \lambda \text{ a.e.}\}$. □

In the previous lemmas and remarks we have established the existence of a unique ACIM for the greedy transformation with arbitrary digits. We have shown that this measure is ergodic, and we have given its support. These results are summarized in the following theorem.

Theorem 2.3.1. *Let $\beta > 1$ and let $A = \{0, a_1, \ldots, a_m\}$ be an allowable digit set for β. If $T : \left[0, \frac{a_m}{\beta-1}\right] \to \left[0, \frac{a_m}{\beta-1}\right]$ is the greedy β-transformation with digits in A, then there exists a unique absolutely continuous invariant measure, that is ergodic. Furthermore, the support of the probability density function h is the interval $[0, y_{i_0}]$, where $i_0 = \min\{i \mid T[0, y_i] \subseteq [0, y_i] \ \lambda \text{ a.e.}\}$.*

The support of the ACIM is the interval $[0, y_{i_0}]$, with i_0 as given in (2.9). Note, however, that by the definition of T, we have $T[0, y_{i_0}) = [0, y_{i_0})$, but the same is not true for any value less than y_{i_0}. Therefore, we can take $[0, y_{i_0})$ as the support of the ACIM. Whenever we consider the restriction of T to the support of the ACIM, we will take this support to be $[0, y_{i_0})$, instead of $[0, y_{i_0}]$. In some situations this will easy the exposition. We can state Theorem 2.3.1 differently, to get the next corollary.

Corollary 2.3.1. *Let $[0, t)$ be the smallest interval such that $T[0, t) \subseteq [0, t)$. The restriction of T to the interval $[0, t)$ admits a unique invariant ergodic measure that is equivalent to Lebesgue measure on this interval.*

It is easy to see that the transformation T, restricted to the support of its ACIM, $[0, y_{i_0})$, is isomorphic to a greedy transformation with arbitrary digits $\bar{T}$, of which the support of the ACIM is equal to the interval $[0, 1)$. The isomorphism θ is given by $\theta : [0, 1) \to [0, y_{i_0}) : x \mapsto y_{i_0} x$ and we have $T \circ \theta = \theta \circ \bar{T}$. Therefore, w.l.o.g., we can assume that $y_{i_0} = 1$.

2.4 The number of digits

In the previous section it is shown that by the results of Li and Yorke in [LY78], on $[0, y_{i_0})$ T has a unique invariant measure that is equivalent to the normalized Lebesgue measure. In general we only have the formulas for the density function of this ACIM given by Kopf ([Kop90]) and Góra ([Gór09]). In the next two sections we discuss two cases for which we can derive a formula for the density that is very similar to (2.2).

In [Wil75], Wilkinson derived a formula for the density of an ACIM for certain piecewise linear transformations. We will use his results in this section. Before we go into more detail, let us give some definitions. We consider our greedy transformation T with $\beta > 1$ and $A = \{0, a_1, \dots, a_m\}$ an allowable digit set for β. By the previous section, we have that the support of the ACIM is given by $[0, y_{i_0}) = [0, 1)$. Suppose that N is the largest index such that $\frac{a_N}{\beta} < y_{i_0}$. Then the points $\frac{a_i}{\beta}$, $i \in \{1, \dots, N\}$, give an interval partition of the interval $[0, y_{i_0})$. Let $\Delta = \{\Delta(0), \dots, \Delta(N)\}$ be the partition of $[0, y_{i_0})$, such that

$$\Delta(0) = \left[0, \frac{a_1}{\beta}\right), \ \Delta(N) = \left[\frac{a_N}{\beta}, y_{i_0}\right) \text{ and } \Delta(i) = \left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\right), \ i \in \{1, \dots, N-1\}.$$

Using Δ and T, we can make the sequence of partitions $\{\Delta^{(n)}\}_{n \geq 1}$ by setting $\Delta^{(n)} = \bigvee_{k=0}^{n-1} T^{-k}\Delta$.

Definition 2.4.1. For $n \geq 1$, let the partition $\Delta^{(n)}$ be given as above. The elements of $\Delta^{(n)}$ are intervals and are called the *fundamental intervals of rank* n. An element $E \in \Delta^{(n)}$ is called *full of rank* n if $\lambda(T^n E) = 1$ and *non-full* otherwise.

Note that Definition 2.4.1 implies that if E is a full fundamental interval of rank n, then $\lambda(E) = 1/\beta^n$. Likewise, if E is a non-full fundamental interval of rank n, then $\lambda(E) < 1/\beta^n$. Definition 2.4.1 is stated for all greedy β-transformations with arbitrary digit sets. We will use this definition in later chapters as well. Now, for $E \in \Delta^{(n)}$, let $\iota(E)$ be the number of non-full fundamental intervals of rank $n+1$, that are contained in E and let

$$\iota_n = \sup_{E \in \Delta^{(n)}} \iota(E).$$

So for each fundamental interval of rank n, we take the number of non-full fundamental intervals of rank $n+1$ it contains. The supremum of these numbers over all the fundamental intervals of rank n is denoted by ι_n. Let ι denote the supremum of these numbers over all ranks, i.e.,

$$\iota = \sup_{n \geq 0} \iota_n,$$

where ι_0 is the number of non-full intervals of rank 1. Applying the results from [Wil75] by Wilkinson to our greedy transformation, would give us a formula for the density of the ACIM in case $\beta > \iota$. We will adapt some of these results to our case and generalize them a bit, so we can say something more. For each $K \geq 0$, let $\bar{\iota}_K = \sup_{n \geq K} \iota_n$. Note that $\iota = \bar{\iota}_0$. Let B_n denote the union of those fundamental intervals of rank n which are full, but which are not a subset of any full fundamental interval of lower rank. We have the following lemma, which is a generalization of Corollary 4.5 in [Wil75].

Lemma 2.4.1. *Let $\bar{\iota}_K$ and B_n be as above and suppose that $\beta > \bar{\iota}_K$ for some $K \geq 0$. Then*
$$\sum_{n=1}^{\infty} \lambda(B_n) = 1.$$

Proof. Consider the support $[0, y_{i_0}) = [0,1)$ and fill it as far as possible with full fundamental intervals of rank 1. Since every non-full fundamental interval of rank 1 has Lebesgue measure smaller than $\frac{1}{\beta}$, the remaining part has Lebesgue measure smaller than $\frac{\iota_0}{\beta}$. Now fill the rest of the interval $[0,1)$ as far as possible with full fundamental intervals of rank 2. The remaining part has Lebesgue measure smaller than $\frac{\iota_0 \cdot \iota_1}{\beta^2}$. If we continue in this manner, after n steps the remaining part will have Lebesgue measure smaller than
$$\iota_0 \iota_1 \cdots \iota_{n-1} \cdot \frac{1}{\beta^n}.$$
And by hypothesis we have
$$\lim_{n\to\infty} \iota_0\iota_1 \cdots \iota_{n-1} \cdot \frac{1}{\beta^n} \leq \iota_0 \iota_1 \cdots \iota_{K-1} \lim_{n\to\infty} \left(\frac{\bar{\iota}_K}{\beta}\right)^n = 0,$$
which completes the proof. □

The next theorem is an adaptation of the formula by Wilkinson and a generalization of Theorem 5.12 in [Wil75]. It gives an explicit expression of the density of the ACIM of the greedy transformation with arbitrary digits under the assumption that $\beta > \bar{\iota}_K$ for some $K \geq 0$. Before we state the theorem, we need the following notation. For all $n \geq 1$, let D_n be the collection of all non-full fundamental intervals of rank n, that are not subsets of any full fundamental interval of lower rank. Let $x \in [0,1)$. Define $\phi_0(x) = 1$ and for $n \geq 1$, let
$$\phi_n(x) = \sum_{E \in D_n} \frac{1}{\beta^n} 1_{T^n E}(x). \tag{2.10}$$

Theorem 2.4.1. *If $\beta > \bar{\iota}_K$ for some $K \geq 0$, then the functions ϕ_n, $n \geq 0$ and ϕ, given by*
$$\phi : [0,1) \to [0,1) : x \mapsto \sum_{n=0}^{\infty} \phi_n(x),$$
are Lebesgue integrable. Moreover, the function h given by
$$h : [0,1) \to [0,1) : x \mapsto \frac{\phi(x)}{\int \phi(x) d\lambda(x)} \tag{2.11}$$
is the density of the ACIM of T.

Proof. The proof follows from Lemma 2.4.1 and a slight adaptation of the corresponding proof in [Wil75]. Wilkinson uses the notion of f-expansion in his proof. The proof here will make use of the Perron-Frobenius equation (1.2).

Let $n \geq 1$. Then

$$\begin{aligned}\int_{[0,1)} \phi_n(x) d\lambda(x) &= \int_{[0,1)} \sum_{E \in D_n} \frac{1}{\beta^n} 1_{T^n E}(x) d\lambda(x) \\ &= \sum_{E \in D_n} \frac{1}{\beta^n} \lambda(T^n E) \leq \left(\frac{\bar{\iota}_n}{\beta} \right)^n .\end{aligned}$$

By the Monotone Convergence Theorem we have

$$\int_{[0,1)} \phi d\lambda \leq 1 + \frac{\bar{\iota}_1}{\beta} + \cdots + \left(\frac{\bar{\iota}_{K-1}}{\beta} \right)^{K-1} + \sum_{n \geq K} \left(\frac{\bar{\iota}_K}{\beta} \right)^n .$$

Since $\bar{\iota}_K < \beta$, the right hand side converges and ϕ is Lebesgue integrable. To show that h is the density of the ACIM, we use (1.2). Therefore, we have to show that

$$\phi(x) = \sum_{i=0}^{N} \frac{1}{\beta} 1_{T\Delta(i)}(x) \phi \left(\frac{x + a_i}{\beta} \right) \qquad \lambda \text{ a.e.}$$

By definition, for all $n \geq 0$,

$$\begin{aligned}\sum_{i=0}^{N} \frac{1}{\beta} 1_{T\Delta(i)}(x) \phi_n \left(\frac{x + a_i}{\beta} \right) &= \sum_{i=0}^{N} \sum_{E \in D_n} \frac{1}{\beta^{n+1}} 1_{T\Delta(i)}(x) 1_{T^n E} \left(\frac{x + a_i}{\beta} \right) \\ &= \sum_{i=0}^{N} \sum_{E \in D_n} \frac{1}{\beta^{n+1}} 1_{\Delta(i) \cap T^n E} \left(\frac{x + a_i}{\beta} \right) .\end{aligned}$$

Each $E \in D_n$ can be divided into fundamental intervals of rank $n+1$, that are either an element of D_{n+1} or are a subset of B_{n+1}. If $E' \subset E$ is a fundamental interval of rank $n+1$, then there is a unique $i \in \{0, \ldots, N\}$, such that $T^n E' \subseteq \Delta(i)$. This is a strict subset if $E' \in D_{n+1}$. If $E' \subseteq B_{n+1}$, then we even know that $T^n E' = \Delta(i)$ and that $\Delta(i)$ is a full fundamental interval of rank 1. Thus,

$$\begin{aligned}\sum_{i=0}^{N} \sum_{E \in D_n} \frac{1}{\beta^{n+1}} 1_{\Delta(i) \cap T^n E} \left(\frac{x + a_i}{\beta} \right) &= \sum_{E \in D_{n+1}} \frac{1}{\beta^{n+1}} 1_{T^{n+1} E}(x) \\ &\quad + \sum_{E \subseteq B_{n+1}, E \in \Delta^{(n+1)}} \frac{1}{\beta^{n+1}} \\ &= \phi_{n+1}(x) + \lambda(B_{n+1}).\end{aligned}$$

By Lemma 2.4.1 we have the result. □

Remark 2.4.1. (i) In the case $K = 0$ Theorem 2.4.1 reduces to the theorem proved by Wilkinson.
(ii) Note the importance of the non-full fundamental intervals in (2.10) and (2.11). These intervals replace the role of 1 for the classical greedy transformation. The non-full fundamental intervals are determined by the orbits of the points y_i. Thus, in order to give the density of the ACIM, we need to know these orbits.

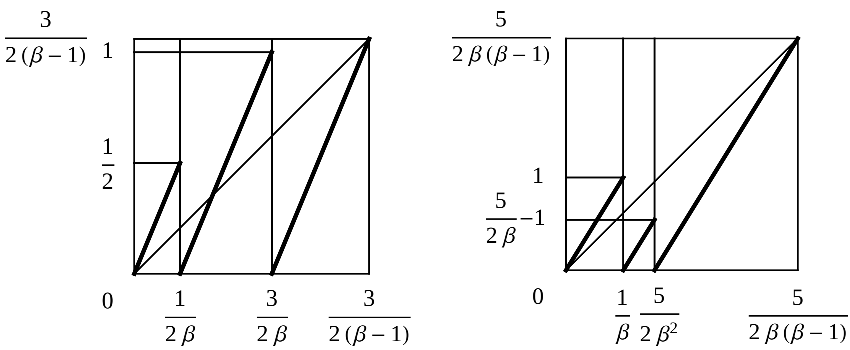
The next theorem states that in the case $m < \beta \leq m+1$, we have that $\beta > \iota$, so we can immediately apply Theorem 2.4.1. In Proposition 2.2.2 it was shown that condition (2.5) implies that $\lceil \beta \rceil \leq m+1$. Thus, the next theorem states that in case the digit set contains the smallest amount of digits possible, the density of the ACIM is given by (2.11).

Theorem 2.4.2. *Let $\beta > 1$ and $A = \{0, a_1, \ldots, a_m\}$ be an allowable digit set, such that $m < \beta \leq m+1$. Let T be the greedy transformation for this β and A. Then the density of the unique absolutely continuous invariant measure for T is given by (2.11).*

Proof. It is enough to show that $\beta > I$. First notice that $\Delta(i_0 - 1)$ is a full fundamental interval, so that we have $\iota_0 \leq N \leq m < \beta$. By the definition of y_{i_0} we have that $\frac{a_{i_0}}{\beta} < y_{i_0}$, which means that $\Delta(i_0 - 1) \neq \Delta(N)$. Let E be a fundamental interval of rank n. Then $T^n E$ is an interval of the form $[0, y) \subseteq [0, y_{i_0})$. This implies that E can contain at most N non-full fundamental intervals of rank $n+1$. So $\iota_n \leq N$ for each $n \geq 1$, which means that $\iota < \beta$, as we wanted. $\square$

We consider two examples. The first one satisfies the condition of Theorem 2.4.2. The second one does not satisfy the condition of this theorem, but in this case we can apply Theorem 2.4.1.

Example 2.4.1. First, let $\beta = 1 + \sqrt{2}$ be the positive solution of the equation $\beta^2 - 2\beta - 1 = 0$ and consider the allowable digit set $A = \{0, \frac{1}{2}, \frac{3}{2}\}$. Since $2 < \beta < 3$, A contains the minimal amount of digits and the condition of Theorem 2.4.2 is satisfied. So the density of the ACIM can be obtained from Theorem 2.4.1. The interval $[0, 1)$ is the support of the invariant measure, see Figure 2.3(a).



(a) $\beta = 1 + \sqrt{2}$ and $A = \{0, \frac{1}{2}, \frac{3}{2}\}$ (b) $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{0, 1, \frac{5}{2\beta}\}$

Figure 2.3 Two examples for which the density of the ACIM is given by (2.11).

The orbits of the points $\frac{1}{2}$ and 1 are as follows.

$$T(\tfrac{1}{2}) = \tfrac{\beta-1}{2}, \quad T^2(\tfrac{1}{2}) = T(\tfrac{\beta-1}{2}) = \tfrac{1}{2\beta}, \quad T^3(\tfrac{1}{2}) = T(\tfrac{1}{2\beta}) = 0,$$

$$T(1) = \beta - \tfrac{3}{2}, \quad T^2(1) = T(\beta - \tfrac{3}{2}) = \tfrac{\beta-1}{2}.$$

So, the orbit of $\frac{1}{2}$ is $\{\frac{1}{2}, \frac{\beta-1}{2}, \frac{1}{2\beta}, 0\}$ and the orbit of 1 is $\{1, \beta - \frac{3}{2}, \frac{\beta-1}{2}, \frac{1}{2\beta}, 0\}$. Then, for $x \in [0,1)$ we have

$$\begin{aligned}\phi(x) &= 1 + \frac{1}{\beta}1_{[0,\frac{1}{2})}(x) + \frac{1}{\beta}1_{[0,\beta-\frac{3}{2})}(x) + \sum_{k=0}^{\infty}\frac{c_{k+2}}{\beta^{k+2}}1_{[0,\frac{1}{2})}(x)\\ &\quad + \sum_{k=0}^{\infty}\frac{c_{k+1}}{\beta^{k+2}}1_{[0,\frac{\beta-1}{2})}(x) + \sum_{k=0}^{\infty}\frac{c_k}{\beta^{k+2}}1_{[0,\frac{1}{2\beta})}(x),\end{aligned}$$

where c_k is the k-th term of the Tribonacci sequence, i.e., $c_k = c_{k-1}+c_{k-2}+c_{k-3}$, starting with $c_0 = 0$, $c_1 = c_2 = 2$. From the recurrence relation one can easily derive the following identity,

$$\sum_{k=0}^{\infty} c_k x^k = \frac{2x}{1 - x - x^2 - x^3}.$$

Using this formula, we get that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{c_{k+2}}{\beta^{k+2}} &= \frac{\frac{2}{\beta}}{1-\frac{1}{\beta}-\frac{1}{\beta^2}-\frac{1}{\beta^3}} - \frac{2}{\beta} = 2 - \frac{1}{\beta},\\ \sum_{k=0}^{\infty}\frac{c_{k+1}}{\beta^{k+2}} &= \frac{1}{\beta}\left[\frac{\frac{2}{\beta}}{1-\frac{1}{\beta}-\frac{1}{\beta^2}-\frac{1}{\beta^3}}\right] = 1,\\ \sum_{k=0}^{\infty}\frac{c_k}{\beta^{k+2}} &= \frac{1}{\beta^2}\left[\frac{\frac{2}{\beta}}{1-\frac{1}{\beta}-\frac{1}{\beta^2}-\frac{1}{\beta^3}}\right] = \frac{1}{\beta}.\end{aligned}$$

So, now

$$\phi(x) = 1 + \frac{1}{\beta}\cdot 1_{[0,\beta-\frac{3}{2})}(x) + 2\cdot 1_{[0,\frac{1}{2})}(x) + 1_{[0,\frac{\beta-1}{2})}(x) + \frac{1}{\beta}\cdot 1_{[0,\frac{1}{2\beta})}(x)$$

and rewriting this, leads to

$$\begin{aligned}\phi(x) &= 2\beta\cdot 1_{[0,\frac{1}{2\beta})}(x) + (\beta+2)\cdot 1_{[\frac{1}{2\beta},\frac{1}{2})}(x) + \beta\cdot 1_{[\frac{1}{2},\frac{\beta-1}{2})}(x)\\ &\quad +(\beta-1)\cdot 1_{[\frac{\beta-1}{2},\beta-\frac{3}{2})}(x) + 1_{[\beta-\frac{3}{2},1)}(x).\end{aligned}$$

Since

$$\int_{[0,1)}\phi(x)d\lambda(x) = \frac{4\beta-2}{\beta},$$

it is easily seen that the measure μ given by

$$\mu(E) = \int_E \frac{\beta}{4\beta-2}\phi(x)d\lambda(x),$$

for every measurable set E, is the unique ACIM for T.

Example 2.4.2. For our second example we let $\beta = \frac{1+\sqrt{5}}{2}$ be the golden mean, i.e., the positive solution of the equation $\beta^2 - \beta - 1 = 0$, and we consider the allowable digit set $A = \{0, 1, \frac{5}{2\beta}\}$. Notice that with this combination of β and A the condition of Theorem 2.4.2 is not satisfied. The support of the invariant measure is given by the interval $[0, 1)$, see Figure 2.3(b). We now look at the orbits of the points 1 and $\frac{5}{2\beta} - 1$.

$$
\begin{array}{llll}
T1 = 1 - \frac{3}{2\beta} = \frac{1}{2\beta^4}, & T^2 1 = \frac{1}{2\beta^3}, & T^3 1 = \frac{1}{2\beta^2}, & T^4 1 = \frac{1}{2\beta}, \\
T^5 1 = \frac{1}{2}, & T^6 1 = \frac{\beta}{2}, & T^7 1 = \frac{\beta^2}{2} - 1 = \frac{1}{2\beta}, &
\end{array}
$$

$$
\begin{array}{lll}
T(\frac{5}{2\beta} - 1) = \frac{3}{2} - \frac{1}{\beta}, & T^2(\frac{5}{2\beta} - 1) = \frac{3}{2\beta} - \frac{1}{2}, & T^3(\frac{5}{2\beta} - 1) = 1 - \frac{1}{2\beta}, \\
T^4(\frac{5}{2\beta} - 1) = \frac{1}{\beta} - \frac{1}{2} = \frac{1}{2\beta}. & &
\end{array}
$$

We see that after the first iteration both orbits never hit $\Delta(2)$ again. This means that the number of non-full fundamental intervals of rank $n \geq 2$ that are not a subset of any full interval of lower rank is at most 1, hence $\bar{\iota}_2 = 1 < \beta$. By Theorem 2.4.1 we can apply formula (2.10) to get

$$
\begin{aligned}
\phi(x) \; = \; & 1 + \frac{1}{\beta} \cdot 1_{[0, \frac{5}{2\beta} - 1)}(x) + \frac{1}{\beta} \cdot 1_{[0, \frac{1}{2\beta^4})}(x) + \frac{1}{\beta^2} \cdot 1_{[0, \frac{3}{2} - \frac{1}{\beta})}(x) + \frac{2}{\beta^3} \cdot 1_{[0, \frac{1}{2\beta^3})}(x) \\
& + \frac{1}{\beta^3} \cdot 1_{[0, \frac{3}{2\beta} - \frac{1}{2})}(x) + \frac{2}{\beta^4} \cdot 1_{[0, \frac{1}{2\beta^2})}(x) + \frac{1}{\beta^4} \cdot 1_{[0, 1 - \frac{1}{2\beta})}(x) \\
& + \left[\frac{1}{\beta^4} + \frac{2}{\beta^7} \sum_{k=0}^{\infty} \frac{1}{\beta^{3k}} \right] 1_{[0, \frac{1}{2\beta})}(x) + \left[\frac{1}{\beta^5} + \frac{2}{\beta^8} \sum_{k=0}^{\infty} \frac{1}{\beta^{3k}} \right] 1_{[0, \frac{1}{2})}(x) \\
& + \left[\frac{1}{\beta^6} + \frac{2}{\beta^9} \sum_{k=0}^{\infty} \frac{1}{\beta^{3k}} \right] 1_{[0, \frac{\beta}{2})}(x).
\end{aligned}
$$

Rewriting this gives

$$
\begin{aligned}
\phi(x) \; = \; & (2\beta + 1) \cdot 1_{[0, \frac{1}{2\beta^4})}(x) + (\beta + 2) \cdot 1_{[\frac{1}{2\beta^4}, \frac{1}{2\beta^3})}(x) + (8 - 3\beta) \cdot 1_{[\frac{1}{2\beta^3}, \frac{1}{2\beta^2})}(x) \\
& + (3\beta - 2) \cdot 1_{[\frac{1}{2\beta^2}, \frac{1}{2\beta})}(x) + (\beta + 1) \cdot 1_{[\frac{1}{2\beta}, \frac{3}{2\beta} - \frac{1}{2})}(x) + (4 - \beta) \cdot 1_{[\frac{3}{2\beta} - \frac{1}{2}, \frac{1}{2})}(x) \\
& + (2\beta - 1) \cdot 1_{[\frac{1}{2}, \frac{5}{2\beta} - 1)}(x) + \beta \cdot 1_{[\frac{5}{2\beta} - 1, 1 - \frac{1}{2\beta})}(x) + (4\beta - 5) \cdot 1_{[1 - \frac{1}{2\beta}, \frac{\beta}{2})}(x) \\
& + (3 - \beta) \cdot 1_{[\frac{\beta}{2}, \frac{3}{2} - \frac{1}{\beta})}(x) + 1_{[\frac{3}{2} - \frac{1}{\beta}, 1)}(x).
\end{aligned}
$$

Furthermore, we have

$$
\int_{[0,1)} \phi(x) d\lambda(x) = \frac{49 - 23\beta}{2\beta^2},
$$

so the density h of the unique ACIM is given by

$$
h(x) = \frac{2\beta^2 \phi(x)}{49 - 23\beta}.
$$

Remark 2.4.2. In [Wil75] more is said about piecewise linear transformations with maximal slope β for which $\iota < \beta$. For example, Wilkinson proved that these transformations are exact and weakly Bernoulli. We can remark that greedy β-transformations with arbitrary digit for which the number of digits $m + 1$ satisfies $m < \beta \leq m + 1$ have these same properties, i.e., they are exact and weakly Bernoulli.

2.5 Making a Markov chain

The condition we impose on the system in this second example is that the endpoints of the transformation must have eventually periodic expansions. What we mean by this is the following. Let T be the greedy transformation for $\beta > 1$ and allowable digit set $A = \{0, a_1, \dots, a_m\}$, restricted to $[0, y_{i_0}) = [0, 1)$ as given in Theorem 2.3.1. Let $N \geq 1$ be the largest index such that $\frac{a_N}{\beta} < 1$. We say that T *has eventually periodic endpoints* if for each $i \in \{1, \dots, N\}$ there exist numbers $u(i)$, $p(i)$, such that $T^{u(i)+p(i)} y_i = T^{u(i)} y_i$. In this case we say that the points y_i have eventually periodic expansions of period $p(i)$.

In [BB85] Byers and Boyarsky proved some nice results about absolutely continuous invariant measures of certain piecewise linear functions, namely the piecewise linear Markov maps. These are defined as follows. Let the numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ give a partition of the interval $[0, 1)$, denoted by α. A map $\tau : [0, 1) \to [0, 1)$ is called a *piecewise linear Markov map* if $\tau|_{(\alpha_j, \alpha_{j+1})}$ is a linear homeomorphism onto some interval $(\alpha_{k(j)}, \alpha_{l(j)})$ for all $j \in \{0, \dots, n-1\}$. If τ is such a map, it induces an $n \times n$ 0-1 matrix $M = M_\tau$ in the following way. The entry m_{jk} equals 1 if $[\alpha_k, \alpha_{k+1}) \subseteq \tau[\alpha_j, \alpha_{j+1})$ and 0 otherwise. The fact that τ is a piecewise linear Markov map guarantees that the nonzero entries in each row are contiguous. In [BB85], Byers and Boyarsky proved the following results.

Let τ be a piecewise linear Markov map, that is expanding and of constant slope and suppose that the 0-1 matrix M it induces is irreducible. Then,

(d1) there exists a unique invariant probability measure μ, that is equivalent to the Lebesgue measure and that maximizes entropy, and,

(d2) the entropy of τ with respect to this measure μ equals $\log \beta$, where β is the spectral radius of M and is also equal to the slope of τ.

A combination of these results, with those found by Parry in [Par64], gives that the invariant measure μ of τ can be found in the following way. Let β be as given by (d2). Let $\mathbf{v} = (v_1, \dots, v_n)$ be the right eigenvector of M, belonging to the eigenvalue β and such that $\sum_{j=1}^n v_j = 1$ and suppose that $\mathbf{u} = (u_1, \dots, u_n)$ is the left eigenvector of M belonging to eigenvalue β and such that $\sum_{j=1}^n u_j v_j = 1$. Then the function

$$\phi : [0, 1) \to [0, 1) : x \mapsto \sum_{j=1}^{n} u_j \cdot 1_{[\alpha_j, \alpha_{j+1})}(x) \tag{2.12}$$

is the density of the unique ACIM for τ that we are looking for.

Now, consider the greedy transformation with arbitrary digits T that has eventually periodic endpoints. Using the orbits of these endpoints, we make a partition α of the interval $[0, 1)$. Consider the set

$$K = \{T^k y_i \,|\, 1 \leq i \leq N, k \geq 0\} \cup \left\{\frac{a_i}{\beta} \,|\, 1 \leq i \leq N\right\} \cup \{0\}.$$

Since all the orbits of the points y_i are finite, this set only contains a finite number of elements, say $n + 1$ elements. So, we can put them in increasing order to get a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$. This gives us the partition α, i.e., $\alpha = \{P_j\}_{j=1}^n$ with $P_j = [\alpha_{j-1}, \alpha_j)$. The next lemma states that T is a piecewise linear Markov map for this partition.

Lemma 2.5.1. *Let T and α be as above. Then T is linear on each of the elements of α and for each $j \in \{1, \dots, n\}$ we have $TP_j = [\alpha_{k(j)}, \alpha_{l(j)})$, for some $\alpha_{k(j)}, \alpha_{l(j)} \in K$.*

Proof. Since all the points $\frac{a_i}{\beta}$ are in K, it is easy to see that T is linear on each element of α. Now fix an element $P_j \in \alpha$. If $\alpha_j = \frac{a_i}{\beta}$ for some $i \in \{1, \dots, N\}$, then $TP_j = [0, T^k y_\ell)$ for some $\ell \in \{1, \dots, N\}$ and $k \geq 0$ and we are done. Suppose $\alpha_j = T^k y_\ell$. If $\alpha_{j+1} = \frac{a_i}{\beta}$, then $T\alpha_j < y_i$, so $TP_j = [T^{k+1} y_\ell, y_i)$. If, on the other hand $\alpha_{j+1} = T^m y_i$, then α_j and α_{j+1} are in the same element of Δ. Thus $T\alpha_j < T\alpha_{j+1}$ and $TP_j = [T^{k+1} y_\ell, T^{m+1} y_i)$. In all cases TP_j is of the desired form. □

The transformation T induces an $n \times n$ 0-1 matrix M_T in the following way. The entry $m_{jk} = 1$ if $P_k \subseteq TP_j$ and $m_{jk} = 0$ otherwise. The following lemma states that each element of α is eventually mapped in each other element of α. This implies that the matrix M_T is irreducible.

Lemma 2.5.2. *For each $j, k \in \{1, \dots, n\}$ there exists a $\ell \geq 0$, such that $P_k \subseteq T^\ell P_j$.*

Proof. Let P_j and P_k be two elements of α. Let μ be the measure given by Theorem 2.3.1 and let h be its density. By (c4) we know that $h > 0$ almost everywhere on P_j and P_k, so P_j and P_k both have strictly positive measure. Moreover, μ is ergodic, so there exists an $\ell \geq 0$, such that

$$\mu(P_j \cap T^{-\ell} P_k) > 0. \tag{2.13}$$

By Lemma 2.5.1, we have that $T^\ell P_j = \bigcup_{i \in \gamma} P_i$ for some index set $\gamma \subseteq \{1, \dots, n\}$. Then (2.13) gives that $\mu(P_k \cap \bigcup_{i \in \gamma} P_i) > 0$, so that $P_k \subseteq T^\ell P_j$. □

By Lemma 2.5.1 the ones in each row of M_T are consecutive and by Lemma 2.5.2, the matrix M_T is irreducible. We have the following theorem.

Theorem 2.5.1. *Let $\beta > 1$ and $A = \{0, a_1, \dots, a_m\}$ be an allowable digit set. Consider T on $[0, 1)$ and suppose that all the endpoints of T have periodic expansions. Let the density of the measure μ_T for all measurable sets $B \subseteq [0, y_{i_0})$ be given by equation (2.12). Then μ_T is the unique ACIM of T and it is the unique measure that maximizes entropy. This entropy is given by $\log \beta$.*

We consider the same two examples as in the previous section.

Example 2.5.1. In the first example, $\beta = 1 + \sqrt{2}$ was the positive solution of the equation $\beta^2 - 2\beta - 1 = 0$, and the allowable digit set considered was $A = \{0, \frac{1}{2}, \frac{3}{2}\}$. We already saw that both endpoints have finite orbits and the partition $\alpha = \{P_j\}_{j=0}^5$, given by these orbits is as follows:

$$P_1 = \left[0, \frac{1}{2\beta}\right), \qquad P_2 = \left[\frac{1}{2\beta}, \frac{1}{2}\right), \qquad P_3 = \left[\frac{1}{2}, \frac{3}{2\beta}\right),$$

$$P_4 = \left[\frac{3}{2\beta}, \frac{\beta-1}{2}\right), \quad P_5 = \left[\frac{\beta-1}{2}, \beta - \frac{3}{2}\right), \quad P_6 = \left[\frac{\beta-3}{2}, 1\right).$$

The matrix $M = M_T$ then becomes

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\mathbf{v} = (v_1, \dots, v_6)$ be the right eigenvector of M with eigenvalue β and such that $\sum_{j=1}^{6} v_j = 1$ and let $\mathbf{u} = (u_1, \dots, u_6)$ be the left eigenvector of M for the eigenvalue β and such that $\sum_{j=1}^{6} u_j v_j = 1$. Then

$$\mathbf{v} = \frac{1}{2\beta^2}(\beta, \beta+1, \beta-1, 1, \beta, 1) \quad \text{and} \quad \mathbf{u} = \frac{\beta}{4\beta-2}(2\beta, \beta+2, \beta, \beta, \beta-1, 1).$$

This means that the invariant probability density for our transformation T is given by

$$\begin{aligned} h(x) \quad = \quad & \frac{\beta}{4\beta-2}\,[2\beta \cdot 1_{[0,\frac{1}{2\beta})}(x) + (\beta+2)\cdot 1_{[\frac{1}{2\beta},\frac{1}{2})}(x) + \beta \cdot 1_{[\frac{1}{2},\frac{\beta-1}{2})}(x) \\ & +(\beta-1)\cdot 1_{[\frac{\beta-1}{2},\beta-\frac{3}{2})}(x) + 1_{[\beta-\frac{3}{2},1)}(x)], \end{aligned}$$

just as we obtained before.

Example 2.5.2. In the second example, $\beta = \frac{1+\sqrt{5}}{2}$ was the golden mean and $A = \{0, 1, \frac{5}{2\beta}\}$ was the allowable digit set. In this case, both the expansion of 1 and that of $\frac{5}{2\beta} - 1$ were eventually periodic and the partition $\alpha = \{P_j\}_{j=1}^{13}$ and 0-1 matrix M they give are the following. For the partition we have

$$P_1 = \left[0, \frac{1}{2\beta^4}\right), \qquad P_2 = \left[\frac{1}{2\beta^4}, \frac{1}{2\beta^3}\right), \qquad P_3 = \left[\frac{1}{2\beta^3}, \frac{1}{2\beta^2}\right),$$

$$P_4 = \left[\frac{1}{2\beta^2}, \frac{1}{2\beta}\right), \qquad P_5 = \left[\frac{1}{2\beta}, \frac{3}{2\beta} - \frac{1}{2}\right), \quad P_6 = \left[\frac{3}{2\beta} - \frac{1}{2}, \frac{1}{2}\right),$$

$$P_7 = \left[\frac{1}{2}, \frac{5}{2\beta} - 1\right), \quad P_8 = \left[\frac{5}{2\beta} - 1, \frac{1}{\beta}\right), \quad P_9 = \left[\frac{1}{\beta}, 1 - \frac{1}{2\beta}\right),$$

$$P_{10} = \left[1 - \frac{1}{2\beta}, \frac{\beta}{2}\right), \quad P_{11} = \left[\frac{\beta}{2}, \frac{3}{2} - \frac{1}{\beta}\right), \quad P_{12} = \left[\frac{3}{2} - \frac{1}{\beta}, \frac{5}{2\beta^2}\right),$$

$$P_{13} = \left[\frac{5}{2\beta^2}, 1\right).$$

And for the matrix,

$$M = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&1\\ 1&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

The right eigenvector, $\mathbf{v}$, with eigenvalue β and such that the sum of its elements equals 1 is

$$\mathbf{v} = \frac{1}{10\beta+6}(\beta, 1, \beta, \beta+1, \beta+1, \beta, 1, \beta, \beta, \beta+1, \beta, \beta, 1)$$

and the left eigenvector, $\mathbf{u}$, belonging to the eigenvalue β and such that the dot product with $\mathbf{v}$ is 1 is

$$\mathbf{u} = \frac{10\beta+6}{29\beta+3}(2\beta+1, 2+\beta, 8-3\beta, 3\beta-2, \beta+1, 4-\beta, 2\beta-1, \beta, \beta, 4\beta-5, 3-\beta, 1, 1).$$

The invariant probability density then is

$$\begin{aligned} h(x) \quad = \quad & \frac{10\beta+6}{29\beta+3}\,[(2\beta+1)\cdot 1_{[0,\frac{1}{2\beta^4})}(x) + (\beta+2)\cdot 1_{[\frac{1}{2\beta^4},\frac{1}{2\beta^3})}(x) \\ & +(8-3\beta)\cdot 1_{[\frac{1}{2\beta^3},\frac{1}{2\beta^2})}(x) + (3\beta-2)\cdot 1_{[\frac{1}{2\beta^2},\frac{1}{2\beta})}(x) + (\beta+1)\cdot 1_{[\frac{1}{2\beta},\frac{3}{2\beta}-\frac{1}{2})}(x) \\ & +(4-\beta)\cdot 1_{[\frac{3}{2\beta}-\frac{1}{2},\frac{1}{2})}(x) + (2\beta-1)\cdot 1_{[\frac{1}{2},\frac{5}{2\beta}-1)}(x) + \beta\cdot 1_{[\frac{5}{2\beta}-1,1-\frac{1}{2\beta})}(x) \\ & +(4\beta-5)\cdot 1_{[1-\frac{1}{2\beta},\frac{\beta}{2})}(x) + (3-\beta)\cdot 1_{[\frac{\beta}{2},\frac{3}{2}-\frac{1}{\beta})}(x) + 1_{[\frac{3}{2}-\frac{1}{\beta},1)}(x)]. \end{aligned}$$

And again, this is equal to the result from the previous paragraph.

Conclusions: In this chapter we have defined the greedy β-transformation with arbitrary digits. We have shown that every such transformation is isomorphic to a similar transformation that has zero as the first digit in the digit set. From that moment on, we have only considered digit sets which are allowable according to Definition 2.2.2. We have proven some properties of this greedy transformation and the expansions it generates. The minimal number of elements that the digit set must contain in order to be allowable is equal to $\lceil\beta\rceil - 1$. The usual ordering on $\mathbb{R}$ respects the lexicographical ordering on the set of digit sequences. The third section of this chapter was devoted to proving the existence of a unique, ergodic, invariant measure for T, that is absolutely continuous with respect to the Lebesgue measure. We found that the support of this ACIM is the interval $[0, y_{i_0})$, where i_0 is given by (2.9).

From [LY82] by Lasota and Yorke, we know that the greedy β-transformation with arbitrary digits is exact.

In the last two sections we have given two specific examples of the greedy transformation in which the density of the ACIM can be explicitly calculated. The first example puts a restraint on the number of digits that can be chosen. If this number is $m+1$ and satisfies $m < \beta \leq m+1$, then the density of the ACIM is given by Wilkinson's formula (2.11). We remarked that in this case the transformation is exact and weak Bernoulli. In the second example we assumed that the endpoints of the transformation have eventually periodic expansions. This allows us to describe the system by a Markov chain. In that case the ACIM is also the measure of maximal entropy (with entropy equal to $\log \beta$) and its density is given by Parry's formula (2.12). In the next chapter we will give a formula resembling (2.2) for the density of the ACIM for all greedy β-transformations with three digits.

What is written in Section 2.2 comes from [DK09b]. Almost everything from Section 2.3 to Section 2.5 can be found in [DK07].

CHAPTER 3

A THREE DIGIT NATURAL EXTENSION

3.1 What is a natural extension?

The advantage of having a transformation to generate expansions, is that we can use ergodic theory as a tool. For that, we need an invariant measure for the transformation under hand. In Section 2.4 and Section 2.5 we have seen two special cases in which we know that the density of the ACIM of the greedy β-transformation is given by (2.11). In this chapter, we will focus on the greedy β-transformation with three digits and establish that the ACIM of this transformation is given by the same formula. Note that the greedy β-transformation is not invertible. A way to obtain an invariant measure is by constructing an invertible dynamical system that contains the dynamics of the original transformation and is minimal from a measure theoretical point of view. Such an invertible system is called a *natural extension*. It is a basic object in ergodic theory and can be used to obtain many properties of the transformation and the expansions it generates. For example, an invariant measure of the transformation can be found through an invariant measure of a natural extension. We will give the definition of a natural extension here.

Definition 3.1.1. Let $(Y, \mathcal{G}, \mu, S)$ be a dynamical system with S a non-invertible transformation. An invertible dynamical system $(X, \mathcal{F}, \nu, U)$ is called a *natural extension* of $(Y, \mathcal{G}, \mu, S)$ if there exist two sets $G \in \mathcal{G}$ and $E \in \mathcal{F}$ and a function $\theta : E \to G$, such that all the following hold.

(ne1) $\mu(Y \setminus G) = \nu(X \setminus E) = 0$,

(ne2) $S(G) \subseteq G$ and $U(E) \subseteq E$,

(ne3) θ is measurable, measure preserving and surjective,

(ne4) $\theta(Uy) = S\theta(y)$ for all $y \in E$, (ne5) $\bigvee_{k=0}^{\infty} U^k\theta^{-1}(\mathcal{G}) = \mathcal{F}$. Recall that $\bigvee_{k=0}^{\infty} U^k\theta^{-1}(\mathcal{G})$ is the smallest σ-algebra containing the σ-algebras $U^k\theta^{-1}(\mathcal{G})$ for all $k \geq 0$.

A function θ that satisfies (ne3) and (ne4) is called a *factor map*.

Example 3.1.1. As an example we will consider the GLS-transformation as defined in Section 1.2.7. For the convenience of the reader, we recall here briefly the definition. Let $\mathcal{D} \subset \mathbb{Z}_+$ be a finite or countable set. A GLS-partition of the interval $[0,1)$ is a set $\mathcal{I} = \{I_k = [\ell_k, r_k) \,|\, k \in \mathcal{D}\}$ of disjoint intervals of increasing length, such that $\lambda(\bigcup_{k\in\mathcal{D}} I_k) = 1$. Set $D = [0,1)\setminus\bigcup_{k\in\mathcal{D}} I_k$. The GLS($\mathcal{I}$)-transformation, $S = S(\mathcal{I}) : [0,1) \to [0,1)$, is defined as follows. For $x \in I_k$, let

$$Sx = \frac{x}{r_k - \ell_k} - \frac{\ell_k}{r_k - \ell_k}$$

and let $Sx = 0$ otherwise. If we set $s(x) = \frac{1}{r_k - \ell_k}$ if $x \in I_k$ and $s(x) = \infty$ otherwise and similarly, $d(x) = \frac{\ell_k}{r_k - \ell_k}$ if $x \in I_k$ and $d(x) = 0$ otherwise, then $Sx = xs(x) - d(x)$. It is very straightforward to give a version of the natural extension of such a transformation. For $n \geq 1$, set $d_n(x) = d(S^{n-1}(x))$ and $s_n(x) = s(S^{n-1}x)$. Then each $x \in [0,1)$ has a GLS($\mathcal{I}$)-expansion, given by

$$x = \sum_{n=1}^{\infty} \frac{d_n}{s_1 s_2 \cdots s_n},$$

and $Sx = s_1 x - d_1$. We would like to define the natural extension of the GLS($\mathcal{I}$)-transformation S on the square $[0,1) \times [0,1)$. To make S invertible, we can define for $y \in [0,1)$,

$$\widehat{S}(x,y) = \left(Sx, \frac{d_1}{s_1} + \frac{y}{s_1}\right).$$

We see this transformation for the example from Section 1.2.7 in Figure 3.1. It turns out that this map satisfies all the conditions. It is measurable with

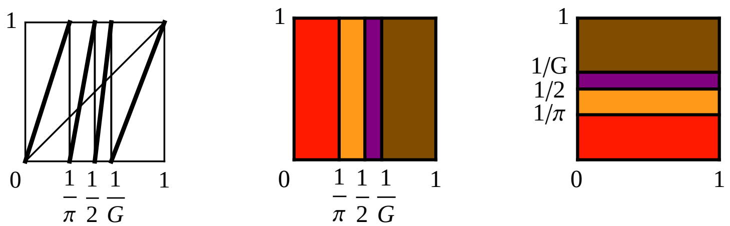


Figure 3.1 The transformation and the natural extension of the GLS-transformation with $\mathcal{D} = \{1,2,3,4\}$ and $I_1 = [1/G, 1)$, $I_2 = [0, 1/\pi)$, $I_3 = [1/\pi, 1/2)$ and $I_4 = [1/2, 1/G)$, where $G = \frac{1+\sqrt{5}}{2}$ is the golden mean.

respect to the σ-algebras $\mathcal{B} \times \mathcal{B}$ and $\mathcal{L} \times \mathcal{L}$, where $\mathcal{B}$ and $\mathcal{L}$ are the Borel and Lebesgue σ-algebras on the unit interval. It is invariant with respect to the Lebesgue measure on $[0,1) \times [0,1)$. The factor map is just the projection onto the first coordinate. If we now consider two-sided sequences of elements from the set $\mathcal{D}$, then the map $\widehat{S}$ would behave like the left-shift on those sequences. For more details and proofs, see [DK02a] by Dajani and Kraaikamp.

The σ-algebras under consideration are in most cases the Lebesgue σ-algebras on appropriate sets. For the verification of (ne5), the next theorem, which can be found in [Roh61] for example, can be useful.

Theorem 3.1.1. *Consider a non-atomic Lebesgue space $(X, \mathcal{F}, \mu)$. Suppose that the set $\{\Delta^{(n)}\}_{n\geq 1}$ is a sequence of partitions of X, such that all the following hold.*

(i) Each element from each partition $\Delta^{(n)}$ is in $\mathcal{F}$.

(ii) For each $n \geq 1$, $\Delta^{(n+1)}$ is a refinement of $\Delta^{(n)}$.

(iii) For any $x_1, x_2 \in X$, there exists a partition $\Delta^{(n)}$, such that x_1 and x_2 are in two distinct elements from $\Delta^{(n)}$.

Then the smallest σ-algebra containing all the sets from $\Delta^{(n)}$ for each n, is denoted by $\bigvee_{n=1}^{\infty} \Delta^{(n)}$ and is equal to $\mathcal{F}$.

Note that for the measurable space $([0,1], \mathcal{L})$ the dyadic intervals satisfy (i)-(iii) of Theorem 3.1.1.

In general, there are many ways to construct a natural extension. In [Roh61], Rohlin gave a canonical way to do this. He also showed that any two natural extensions of the same dynamical system are isomorphic. So, we can speak of the natural extension of a dynamical system. If we want to use the natural extension to prove properties of a transformation, then the success of the whole process depends heavily on the version of the natural extension that we choose to work with. Some of the properties that go over from the natural extension to the original system are the following.

Proposition 3.1.1. *If the transformation from a natural extension is ergodic/weakly mixing/strongly mixing/weakly Bernoulli/Bernoulli, then the original transformation is ergodic/weakly mixing/strongly mixing/weakly Bernoulli/Bernoulli.*

A proof of this proposition can be found in any standard text book on ergodic theory. See for example [Wal82] by Walters. Theorem 3.1.2 gives another important result related to natural extensions. For that we need the following definition. Recall that in Definition 1.2.13 we defined exactness for transformations.

Definition 3.1.2. An invertible, measurable transformation $\widehat{T}$, defined on a probability space $(X, \mathcal{F}, \mu)$ is called a *K-automorphism* if it is the natural extension transformation of an exact transformation.

The following theorem follows from results by Rychlik, which can be found in [Ryc83]. In [Ryc83], Rychlik considered a specific class of transformations and the transformations that we consider here, i.e., surjective, piecewise linear, expanding maps with constant slope, defined from an interval to itself, fall into that category. The results from [Ryc83] are of a far more general nature than Theorem 3.1.2 below, but we will state here only what is needed for this chapter. For more information or the precise result, the reader is referred to [Ryc83].

Theorem 3.1.2 (Rychlik, [Ryc83])**.** *Consider a probability space $(X, \mathcal{F}, \mu)$ and a measure preserving transformation T. Suppose that T is a piecewise linear, expanding map with constant slope, defined from an interval to itself. Let $\widehat{T}$ be a natural extension transformation for T. If $\widehat{T}$ is a K-automorphism, then $\widehat{T}$ is weakly Bernoulli and thus also T is weakly Bernoulli.*

In this chapter we will study the greedy β-transformation with three digits and give an expression for the ACIM by defining the natural extension. We will first give conditions under which we can directly see that the density of the ACIM of this transformation is the density from (2.11). This happens in Section 3.2. In Section 3.3 we consider a specific example of a greedy β-transformation and give three versions of the natural extension of its dynamical system. Using this example as a starting point, we then define one version of the natural extension for general greedy β-transformations with three digits. This version will allow us to obtain an expression for the ACIM of the transformation by projecting onto the first coordinate. In all cases the density of the ACIM will turn out to be the density from (2.11). For the classical greedy β-transformation, a version of the natural extension is given in [DKS96] by Dajani, Kraaikamp and Solomyak and also in [BY00] by Brown and Yin. The definition from Section 3.5 is based on the version given in [DKS96]. The ACIM μ for the greedy β-transformation will be defined by "pulling back" the invariant measure on the natural extension. We obtain that the transformation is exact and weakly Bernoulli. Of course, we already know that the transformation is exact by [LY82] from Lasota and Yorke. To illustrate the general construction of the natural extension, we give an example of a specific greedy β-transformation with three digits in the last section.

3.2 Three is a company

Let $\beta > 1$ be a real number and let $u > 1$ be any real number such that the set $A = \{0, 1, u\}$ is an allowable digit set for β according to Definition 2.2.2. From Proposition 2.2.2 we know that A can only be allowable for β with $1 < \beta < 3$. Note that every greedy β-transformation with three digits is isomorphic to a greedy β-transformation for the same β, but with digit set $\{0, 1, u\}$. So, w.l.o.g., in this chapter we will only consider greedy β-transformations that have $1 < \beta < 3$ and allowable digit sets A of the form $A = \{0, 1, u\}$. We will first give some conditions under which we can directly see that the density from (2.11) is the density of the ACIM.

If T is a greedy β-transformation with $1 < \beta < 3$ and allowable digit set

$A = \{0, 1, u\}$, then the transformation is given by:

$$Tx = \begin{cases} \beta x, & \text{if } x \in \left[0, \dfrac{1}{\beta}\right), \\ \beta x - 1, & \text{if } x \in \left[\dfrac{1}{\beta}, \dfrac{u}{\beta}\right), \\ \beta x - u, & \text{if } x \in \left[\dfrac{u}{\beta}, \dfrac{u}{\beta - 1}\right]. \end{cases}$$

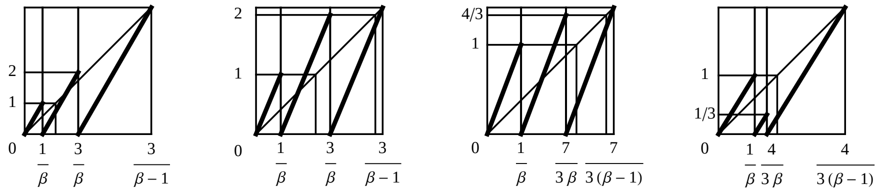
For $x \in \left[0, \frac{u}{\beta-1}\right]$, let $(b_n(x))_{n\geq 1}$ denote the digit sequence of x generated by T. Before we can give the density function of the ACIM, we need to find its support. From (2.9) we know that this support is either the interval $[0, 1)$ or the interval $[0, u - 1)$. The following situations can occur.

- Suppose first that $\beta < u$. Either $T1 \leq 1$ or $T1 > 1$.
  - If $T1 \leq 1$, then the support is $[0, 1)$ and since $T1 = \beta - 1$, we get that $\beta \leq 2$. The transformation T on the interval $[0, 1)$ is then isomorphic to the classical greedy β-transformation, T_β, for the same β. Figure 3.2(a) is an example of this.
  - If $T1 > 1$, then the support of the invariant measure is the other interval, $[0, u-1)$, and we can deduce that $\beta > 2$. So, Theorem 2.4.2 applies and the density for the invariant measure is given by (2.11). In Figure 3.2(b) we see an example.
- Suppose that $u \leq \beta$. Either $u > 2$ or $u \leq 2$.
  - If $u > 2$, then the support of the ACIM is $[0, u - 1)$ and we have that $2 < u \leq \beta$. So again $\beta > 2$ and the density from equation (2.10) is the density of the invariant measure we are looking for. See Figure 3.2(c) for an example.
  - If $u = 2$, then the support is $[0, 1) = [0, u - 1)$ and $\beta \geq 2$. The transformation in this case is isomorphic to the classical greedy transformation for the same β. Similarly, if $u = \beta < 2$, then the support of the ACIM is $[0, 1)$. The transformation T is again isomorphic to the classical greedy β-transformation for the same β.
  - Lastly, suppose that $u = \beta - 1$. Then $\beta > 2$, since $u > 1$ and thus the density from (2.11) is the density for the ACIM.

The only situation we did not yet consider, is when $u < \min\{\beta, 2\}$ and $u \neq \beta - 1$. To this, the rest of this chapter is dedicated. Of course, since A is allowable and $u \neq \beta - 1$, we also have $u > \beta - 1$. So, from now on, suppose that T is a greedy β-transformation with an allowable digit set $A = \{0, 1, u\}$, that satisfies the following condition.

$$\max\{\beta - 1, 1\} < u < \min\{2, \beta\}. \tag{3.1}$$

The support of the ACIM of T is the interval $[0, 1)$. Figure 3.2(d) gives an example of a transformation that satisfies these conditions.



(a) $\beta = \sqrt{3}$, $A = \{0, 1, 3\}$ (b) $\beta = 1 + \sqrt{2}$, $A = \{0, 1, 3\}$ (c) $\beta = \sqrt{7}$, $A = \{0, 1, \frac{7}{3}\}$ (d) $\beta = \frac{1+\sqrt{5}}{2}$, $A = \{0, 1, \frac{4}{3}\}$

Figure 3.2 Examples of the four possibilities for the support of the ACIM of the greedy β-transformation with three digits.

Note that we do not assume that $\beta \leq 2$, although we already know that the density from (2.11) is the density of the ACIM for all $2 < \beta < 3$. The reason is that the construction of the natural extension that we will give, is also valid for $2 < \beta < 3$.

Remark 3.2.1. Observe that, if $1 < \beta \leq 2$, then $u < \beta$ implies condition (3.1). So, if $1 < \beta \leq 2$ and a digit set $A = \{0, 1, u\}$ satisfies $u < \beta$, then A is an allowable digit set.

Some of the proofs from Section 3.3 and Section 3.4 use properties of the fundamental intervals, defined in Definition 2.4.1. We will recall their definition here and discuss some of their structure, for future reference. For the greedy β-transformation with digit set $A = \{0, 1, u\}$ and β satisfying (3.1), the fundamental intervals of rank 1 are given by

$$\Delta(0) = \left[0, \frac{1}{\beta}\right), \quad \Delta(1) = \left[\frac{1}{\beta}, \frac{u}{\beta}\right) \quad \text{and} \quad \Delta(u) = \left[\frac{u}{\beta}, 1\right).$$

They give a partition of $[0, 1)$. For each $n \geq 1$, let $\Delta^{(n)}$ denote the set of all fundamental intervals of rank n. Each $E \in \Delta^{(n)}$ will have the form

$$\Delta(b_1) \cap T^{-1}\Delta(b_2) \cap \cdots \cap T^{-(n-1)}\Delta(b_n)$$

for some $b_1, b_2, \ldots, b_n \in \{0, 1, u\}$. We will therefore denote them by $\Delta(b_1 \cdots b_n)$. Note that a fundamental interval of rank n specifies the first n digits of the expansion of the elements it contains. For full fundamental intervals, we have the following obvious lemma.

Lemma 3.2.1. *Let $\Delta(d_1 \cdots d_p)$ and $\Delta(b_1 \cdots b_q)$ be two full fundamental intervals of rank p and q respectively. Then the set $\Delta(d_1 \cdots d_p b_1 \cdots b_q)$ is a full fundamental interval of rank $p + q$.*

Recall that, if $\Delta(b_1 \cdots b_n)$ is a full fundamental interval of rank n, then

$$\lambda(\Delta(b_1 \cdots b_n)) = \frac{1}{\beta^n}, \tag{3.2}$$

and if $\Delta(b_1 \cdots b_n)$ is a non-full fundamental interval of rank n, then

$$\lambda(\Delta(b_1 \cdots b_n)) < \frac{1}{\beta^n}. \tag{3.3}$$

In the present set-up, $\Delta(0)$ is a full fundamental interval of rank 1. The other two fundamental intervals of rank 1 are non-full.

3.3 Three times is a charm

In this section we will consider a specific example of a greedy β-transformation with arbitrary digits. It has β equal to the golden mean and the digit set is $\{0,1,\frac{3}{2}\}$. We will define three versions of its natural extension. All of these versions will prove to be invariant with respect to a normalized Lebesgue measure. This will allow us to give an invariant measure for the original transformation, equivalent to the Lebesgue measure, by simply projecting the measure of the natural extension to the first coordinate. By looking at this specific example in detail, we can identify properties of the transformation that are important for a general construction. This general construction will be given in Section 3.5 and is a generalization of the first version from this section. The second version of the natural extension is a planar one and therefore is perhaps more easy to visualize. The construction of this planar version is so, that it might be possible to generalize it for a larger class of transformations. However, this is not straightforward and we do not investigate this further. The last version we consider is also a planar one. The domain of the natural extension is simpler, but its construction depends heavily on the properties of the chosen β and digit set.

Let β be the golden mean, i.e., the positive solution of the equation $x^2 - x - 1 = 0$. Take $A = \{0,1,\frac{3}{2}\}$. Then $y_1 = 1$ and $y_2 = \frac{1}{2}$. Since $T1 = \beta - \frac{3}{2} < 1$, by Theorem 2.3.1 we know that the support of the ACIM of T is the interval $[0,1)$. Consider the partition $\Delta = \{\Delta(0),\Delta(1),\Delta(3/2)\}$ of the interval $[0,1)$, given by $\Delta(0) = \left[0,\frac{1}{\beta}\right)$, $\Delta(1) = \left[\frac{1}{\beta},\frac{3}{2\beta}\right)$ and $\Delta(3/2) = \left[\frac{3}{2\beta},1\right)$. The transformation T is defined by $Tx = \beta x - i$ if $x \in \Delta(i)$. It is shown on $[0,1)$ in Figure 3.3. To each $x \in [0,1)$ we assign a digit sequence $(b_n(x))_{n\geq 1}$ as usual. Let

$$b_1(x) = \begin{cases} 0, & \text{if } x \in \left[0,\dfrac{1}{\beta}\right), \\ 1, & \text{if } x \in \left[\dfrac{1}{\beta},\dfrac{3}{2\beta}\right), \\ \dfrac{3}{2}, & \text{if } x \in \left[\dfrac{3}{2\beta},1\right), \end{cases}$$

and for $n \geq 1$, set $b_n(x) = b_1(T^{n-1}x)$. Two expansions that will play an important role in what follows are the expansions of the points $\frac{1}{2}$ and $\frac{1}{2\beta^3}$. Note that $\frac{1}{2\beta^3} = \beta - \frac{3}{2}$ would be the image of 1 under T if we would have considered T on the closed interval $[0,1]$. We have

$$\frac{1}{2} = \sum_{k=1}^{\infty}\frac{d_k^{(1)}}{\beta^k} = \frac{1}{\beta^2}+\frac{1}{\beta^5}+\frac{1}{\beta^8}+\frac{1}{\beta^{11}}+\cdots = .01(001)^{\omega}, \tag{3.4}$$

$$\frac{1}{2\beta^3} = \sum_{k=1}^{\infty}\frac{d_k^{(3/2)}}{\beta^k} = \frac{1}{\beta^4}+\frac{1}{\beta^7}+\frac{1}{\beta^{10}}+\cdots = .00(001)^{\omega}. \tag{3.5}$$

In Figure 3.3, you can also see the orbits of the points $\frac{1}{2}$ and $\frac{1}{2\beta^3}$.

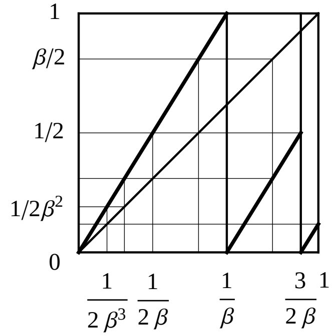


Figure 3.3 The transformation T and the orbits of $\frac{1}{2}$ and $\frac{1}{2\beta^3}$.

Now, consider the fundamental intervals of T. By looking at Figure 3.3, we see that after two steps, $\Delta(1)$ splits into a full and a non-full piece, i.e., $\Delta(100)$ is full and $\Delta(101)$ is non-full. The same holds for $\Delta(\frac{3}{2})$ after five steps, i.e., $\Delta(\frac{3}{2}00000)$ is full and $\Delta(\frac{3}{2}00001)$ is non-full. After that, each remaining non-full fundamental interval splits into a full and a non-full piece after each three steps: $\Delta(\frac{3}{2}00001000)$ is full, $\Delta(\frac{3}{2}00001001)$ is non-full, $\Delta(101000)$ is full, $\Delta(101001)$ is non-full, etc. This is also given by the periodicity of the expansions of $\frac{1}{2}$ and $\frac{1}{2\beta^3}$. From the next lemma, it follows that the full fundamental intervals generate the Borel σ-algebra on $[0,1)$.

Lemma 3.3.1. *For each $n \geq 1$, let B_n be the union of those full fundamental intervals of rank n that are not subsets of any full fundamental interval of lower rank, as in Section 2.3. Then $\sum_{n=1}^{\infty} \lambda(B_n) = 1$.*

Proof. Notice that $B_1 = \Delta(0)$, $B_3 = \Delta(100)$ and for $k \geq 1$, B_{3k+1} is the union of two intervals. For all the other values of n, $B_n = \emptyset$. So

$$\sum_{n=1}^{\infty} \lambda(B_n) = \frac{1}{\beta} + \frac{1}{\beta^3} + \sum_{k=1}^{\infty} \frac{2}{\beta^{3(k+1)}} = \frac{1}{\beta} + \frac{1}{\beta^3} + \frac{2}{\beta^3}\left[\frac{1}{1-1/\beta^3} - 1\right] = 1. \qquad \square$$

Remark 3.3.1. The fact that $\Delta(0)$ is a full fundamental interval of rank 1 allows us to construct full fundamental intervals of arbitrary small Lebesgue measure. This together with the previous lemma guarantees that we can write each interval in $[0,1)$ as a countable union of these full intervals. Thus, the full fundamental intervals generate the Borel σ-algebra on $[0,1)$.

3.3.1 *Two rows of rectangles*

To find an expression for the T-invariant measure, μ, absolutely continuous with respect to the Lebesgue measure, we define three versions of the natural extension of the dynamical system $([0,1), \mathcal{B}, \mu, T)$. For the definition of the first version, we use a subcollection of the collection of fundamental intervals.

For $n \geq 1$, let D_n denote the collection of all non-full fundamental intervals of rank n that are not a subset of any full fundamental interval of lower rank, as in Section 2.3. The elements of D_n can be explicitly given as follows. $D_1 = \{\Delta(1), \Delta(\frac{3}{2})\}$, $D_2 = \{\Delta(10), \Delta(\frac{3}{2}0)\}$ and for $k \geq 1$,

$$\begin{aligned}
D_{3k} &= \Big\{\Delta(101\underbrace{001\cdots001}_{k-1\text{ times}}), \Delta\Big(\frac{3}{2}00\underbrace{001\cdots001}_{k-1\text{ times}}\Big)\Big\},\\
D_{3k+1} &= \Big\{\Delta(101\underbrace{001\cdots001}_{k-1\text{ times}}0), \Delta\Big(\frac{3}{2}00\underbrace{001\cdots001}_{k-1\text{ times}}0\Big)\Big\},\\
D_{3k+2} &= \Big\{\Delta(101\underbrace{001\cdots001}_{k-1\text{ times}}00), \Delta\Big(\frac{3}{2}00\underbrace{001\cdots001}_{k-1\text{ times}}00\Big)\Big\}.
\end{aligned}$$

For each element $\Delta(b_1\cdots b_n)$ of D_n, $T^n\Delta(b_1\cdots b_n)$ is one of the intervals from the set

$$\left\{\left[0, \frac{1}{2\beta^3}\right), \quad \left[0, \frac{1}{2\beta^2}\right), \quad \left[0, \frac{1}{2\beta}\right), \quad \left[0, \frac{1}{2}\right), \quad \left[0, \frac{\beta}{2}\right)\right\}.$$

The domain of the natural extension will consist of two sequences of sets $\{R_{(1,n)}\}_{n\geq 1}$ and $\{R_{(3/2,n)}\}_{n\geq 1}$, that represent the images of the elements of D_n under T^n. We will order them in two rows by assigning two extra parameters to each rectangle. Let $R_0 = [0,1) \times [0,1)$ and for each $n \geq 1$, $i \in \{1, \frac{3}{2}\}$ define the sets

$$R_{(i,n)} = T^n\Delta(id_1^{(i)}\cdots d_{n-1}^{(i)}) \times \Delta(\underbrace{0\cdots0}_{n\text{ times}}),$$

where the digits $d_n^{(i)}$ are the elements of the digit sequences of $\frac{1}{2}$ and $\frac{1}{2\beta^3}$ as given in (3.4) and (3.5). Then, set

$$R = R_0 \times \{0\} \times \{0\} \cup \bigcup_{n=1}^{\infty}(R_{(1,n)} \times \{1\} \times \{n\} \cup R_{(3/2,n)} \times \{3/2\} \times \{n\}).$$

Let $\mathcal{B}_0$ denote the Borel σ-algebra on R_0 and on each of the rectangles $R_{(i,n)}$, let $\mathcal{B}_{(i,n)}$ denote the Borel σ-algebra defined on it. We define the σ-algebra $\mathcal{R}$ on R to be the disjoint union of all the σ-algebras $\mathcal{B}_0$ and $\mathcal{B}_{(i,n)}$. Let $\bar{\lambda}$ be the measure on $(R, \mathcal{R})$, given by the Lebesgue measure on each rectangle. Then $\bar{\lambda}(R) = 8 - \frac{7}{2}\beta$. If we set $\nu = \frac{2}{16-7\beta}\bar{\lambda}$, then $(R, \mathcal{R}, \nu)$ will be a probability space.

We define the transformation $\widehat{T}$ piecewise on the sets of R. If $(x, y) \in R_0$, let

$$\widehat{T}(x, y, 0, 0) = \begin{cases} (Tx, \frac{y}{\beta}, 0, 0), & \text{if } x \in \Delta(0), \\ (Tx, \frac{y}{\beta}, i, 1), & \text{if } x \in \Delta(i),\ i \in \{1, \frac{3}{2}\}, \end{cases}$$

and for $(x, y) \in R_{(i,n)}$, let

$$\widehat{T}(x, y, i, n) = \begin{cases} (Tx, y^{(i)}, 0, 0), & \text{if } \Delta(id_1^{(i)}\cdots d_{n-1}^{(i)}0) \text{ is full} \\ & \quad \text{and } x \in \Delta(0), \\ (Tx, \frac{y}{\beta}, i, n+1), & \text{if } \Delta(id_1^{(i)}\cdots d_{n-1}^{(i)}0) \\ & \quad \text{is non-full or } x \notin \Delta(0), \end{cases}$$

where

$$y^{(i)} = \frac{i}{\beta} + \frac{d_1^{(i)}}{\beta^2} + \frac{d_2^{(i)}}{\beta^3} + \cdots + \frac{d_{n-1}^{(i)}}{\beta^n} + \frac{y}{\beta}.$$

Figure 3.4 shows the space R.

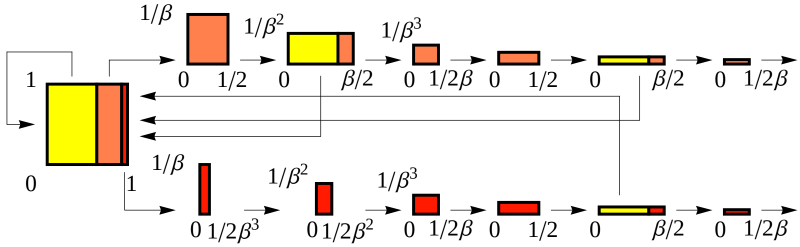


Figure 3.4 The space R consists of all these rectangles. The arrows indicate how $\widehat{T}$ moves the rectangles.

Remark 3.3.2. Note that for $k \geq 1$, $\widehat{T}$ maps all rectangles $R_{(1,n)}$ for which $n \neq 3k-1$ and all rectangles $R_{(3/2,n)}$ for which $n \neq 3k+2$ bijectively onto $R_{(1,n+1)}$ and $R_{(3/2,n+1)}$ respectively. The rectangles $R_{(1,3k-1)}$ and $R_{(3/2,3k+2)}$ are partly mapped onto $R_{(1,3k)}$ and $R_{(3/2,3k+3)}$ respectively and partly into R_0. From Lemma 3.3.1 it follows that $\widehat{T}$ is bijective.

Let $\pi : R \to [0,1)$ be the projection onto the first coordinate. Define the measure μ on $([0,1),\mathcal{B})$ by $\mu = \nu \circ \pi^{-1}$. To show that $(R,\mathcal{B},\nu,\widehat{T})$ is a version of the natural extension of $([0,1),\mathcal{B},\mu,T)$ with π as a factor map, we need to prove (ne1)-(ne5). It is clear that π is surjective and measurable and that $T\circ\pi = \pi\circ\widehat{T}$. Since $\widehat{T}$ expands by a factor β in the first coordinate and contracts by a factor β in the second coordinate, it is also clear that $\widehat{T}$ is invariant with respect to the measure ν. Then $\mu = \nu \circ \pi^{-1}$ defines a T-invariant probability measure on $([0,1),\mathcal{B})$, that is equivalent to the Lebesgue measure, and π is a measure preserving map. This shows (ne1) to (ne4), so that leaves only (ne5). To prove (ne5), we need to have a closer look at the structure of the full fundamental intervals and we will introduce some more notation.

For a full fundamental interval, $\Delta(b_1 \cdots b_n)$, we can make a subdivision of the block of digits $b_1 \cdots b_n$ into several subblocks, each of which corresponds to a full fundamental interval itself. To make this subdivision more precise, we need the notion of return time. For points $(x,y) \in R_0$ define the *first return time to* R_0 by

$$r_1(x,y) = \inf\{j \geq 1 \,|\, \widehat{T}^j(x,y,0,0) \in R_0 \times \{0\} \times \{0\}\}$$

and for $k > 1$, let the *k-th return time to* R_0 be given recursively by

$$r_k(x,y) = \inf\{j > r_{k-1}(x,y) \,|\, \widehat{T}^j(x,y,0,0) \in R_0 \times \{0\} \times \{0\}\}.$$

By the Poincaré Recurrence Theorem, we have $r_k(x,y) < \infty$ for almost all $(x,y) \in R_0$. Notice that this notion depends only on x, i.e., for all $(x,y),(x,y') \in$

R_0 and all $k \geq 1$, $r_k(x,y) = r_k(x,y')$. So we can write $r_k(x)$ instead of $r_k(x,y)$. Let $\Delta(b_1 \cdots b_n) \in \Delta^{(n)}$ be a full fundamental interval. Then for all $m \leq n$, $\widehat{T}^m$ maps the whole set $\Delta(b_1 \cdots b_n) \times [0,1) \subseteq R_0$ to the same rectangle in R. So the first several return times to R_0 are equal for all $x \in \Delta(b_1 \cdots b_n)$. Hence, there is a maximal number $M \geq 1$ and there are numbers r_k, $1 \leq k \leq M$, such that $r_k = r_k(x)$ for all $x \in \Delta(b_1 \cdots b_n)$ and $r_M = n$. Put $r_0 = 0$. We can also obtain the numbers r_k inductively as follows. Let

$$r_1 = \inf\{j \geq 1 \mid T^j \Delta(b_1 \cdots b_j) = [0,1)\}$$

and if $r_1, \ldots, r_{k-1}$ are already known, let

$$r_k = \inf\{j > r_{k-1} \mid T^j \Delta(b_{r_{k-1}+1} \cdots b_j) = [0,1)\}.$$

For $1 \leq k \leq M$, let the subblock C_k be

$$C_k = b_{r_{k-1}+1} \cdots b_{r_k}.$$

Let $|C_k|$ denotes number of digits in the block C_k. For full fundamental intervals $\Delta(b_1 \cdots b_n)$, the subblocks C_k have the following properties.

(p1) For all $1 \leq k \leq M$, $|C_k| = r_k - r_{k-1}$.

(p2) If $b_{r_k+1} = 0$, then $r_{k+1} = r_k + 1$. This means that if a subblock begins with the digit 0, then 0 is the only digit in this subblock. So, C_{k+1} consists just of the digit 0.

(p3) If $b_{r_k+1} = i \in \{1, u = \frac{3}{2}\}$, then the block C_{k+1} is equal to i followed by the first part of the expansion of 1 if $i = 1$ and that of $\beta - u = 1/2\beta^3$ if $i = u = \frac{3}{2}$. So $C_{k+1} = i d_1^{(i)} \cdots d_{|C_{k+1}|-1}^{(i)}$.

(p4) For all $1 \leq k \leq M$, $\Delta(C_k)$ is a full fundamental interval of rank $|C_k|$.

The above procedure gives for each full fundamental interval $\Delta(b_1 \cdots b_n)$, a subdivision of the block of digits $b_1 \cdots b_n$ into subblocks $C_1, \ldots, C_M$, such that $\Delta(C_k)$ is a full fundamental interval of rank $|C_k|$ and $\Delta(b_1 \cdots b_n) = \Delta(C_1 \cdots C_M)$. The next lemma is the last step in proving that the system $(R, \mathcal{R}, \nu, \widehat{T})$ is a version of the natural extension.

Lemma 3.3.2. *The σ-algebra $\mathcal{R}$ on R and the σ-algebra $\bigvee_{n=0}^{\infty} \widehat{T}^n \pi^{-1}(\mathcal{B})$ are equal.*

Proof. First notice that by Lemma 3.3.1, each of the σ-algebras $\mathcal{B}_{(i,n)}$ is generated by the direct products of the full fundamental intervals, contained in the rectangle $R_{(i,n)}$. Also, $\mathcal{B}_0$ is generated by the direct products of the full fundamental intervals. It is clear that $\bigvee_{n=0}^{\infty} \widehat{T}^n \pi^{-1}(\mathcal{B}) \subseteq \mathcal{R}$. For the other inclusion, first take a generating rectangle in R_0:

$$\Delta(d_1 \cdots d_p) \times \Delta(b_1 \cdots b_q) \times \{0\} \times \{0\},$$

where $\Delta(d_1 \cdots d_p)$ and $\Delta(b_1 \cdots b_q)$ are full fundamental intervals. For the set $\Delta(b_1 \cdots b_q)$ construct the subblocks $C_1, \ldots, C_M$. By Lemma 3.2.1, the set

$$\Delta(C_M C_{M-1} \cdots C_1 d_1 \cdots d_p)$$

is a full fundamental interval of rank $p+q$. Then

$$\begin{aligned}\pi^{-1}(\Delta(C_M C_{M-1}\cdots C_1 d_1\cdots d_p)) &\cap (R_0\times\{0\}\times\{0\}) \\ &= \Delta(C_M C_{M-1}\cdots C_1 d_1\cdots d_p)\times[0,1)\times\{0\}\times\{0\}.\end{aligned}$$

Since $\Delta(b_1\cdots b_q)$ is a full fundamental interval, it can be proven by induction that for all $1\le k\le q$, $T^k\Delta(b_1\cdots b_q)=\Delta(b_k\cdots b_q)$. This, together with the definitions of the blocks C_k and the transformation $\widehat{T}$ leads to

$$\begin{aligned}&\widehat{T}^q(\pi^{-1}(\Delta(C_M C_{M-1}\cdots C_1 d_1\cdots d_p))\cap R_0\times\{0\}\times\{0\}) \\ &\Delta(d_1\cdots d_p)\times\Delta(b_1\cdots b_q)\times\{0\}\times\{0\}\subseteq\bigvee_{n=0}^{\infty}\widehat{T}^n\pi^{-1}(\mathcal{B}).\end{aligned}$$

Now let $\Delta(d_1\cdots d_p)\times\Delta(b_1\cdots b_q)\times\{i\}\times\{n\}$ be a generating rectangle for $\mathcal{B}_{(i,n)}$, $i\in\{1,\frac{3}{2}\}$ and $n\ge 1$. So $\Delta(d_1\cdots d_p)$ and $\Delta(b_1\cdots b_q)$ are again full fundamental intervals. Notice that

$$\Delta(b_1\cdots b_q)\subseteq\Delta(\underbrace{0\cdots 0}_{n\text{ times}}),$$

which means that $q\ge n$. For all $1\le k\le n$, $b_k=0$ and $r_k=k$. So, if we divide $b_1\cdots b_q$ into subblocks C_i as before, we get that $C_1=C_2=\cdots=C_n=0$, that $M\ge n$ and that $|C_{n+1}|+\cdots+|C_M|=q-n$. Consider the set $C=\Delta(C_M C_{M-1}\cdots C_{n+1} i d_1^{(i)}\cdots d_{n-1}^{(i)} d_1\cdots d_p)$. We will show the following.

Claim: The set C is a fundamental interval of rank $p+q$ and $T^qC=\Delta(d_1\cdots d_p)$.
Proof of claim: First notice that

$$C=\Delta(C_M C_{M-1}\cdots C_{n+1})\cap T^{n-q}\Delta(id_1^{(i)}\cdots d_{n-1}^{(i)})\cap T^{-q}\Delta(d_1\cdots d_p).$$

So obviously,

$$T^qC\subseteq T^q\Delta(C_M C_{M-1}\cdots C_{n+1})\cap T^n\Delta(id_1^{(i)}\cdots d_{n-1}^{(i)})\cap\Delta(d_1\cdots d_p).$$

By Lemma 3.2.1, $\Delta(C_M C_{M-1}\cdots C_{n+1})$ is a full fundamental interval of rank $q-n$, so $T^q\Delta(C_M C_{M-1}\cdots C_{n+1})=[0,1)$. Now, by the definition of $R_{(i,n)}$ we have that

$$\Delta(d_1\cdots d_p)\subseteq T^n\Delta(id_1^{(i)}\cdots d_{n-1}^{(i)}), \tag{3.6}$$

and thus $T^qC\subseteq\Delta(d_1\cdots d_p)$.
For the other inclusion, let $z\in\Delta(d_1\cdots d_p)$. Then there is an element y in $\Delta(id_1^{(i)}\cdots d_{n-1}^{(i)})$, such that $T^ny=z$. Since $T^{q-n}\Delta(C_M C_{M-1}\cdots C_{n+1})=[0,1)$, there is an $x\in\Delta(C_M C_{M-1}\cdots C_{n+1})$ with $T^{q-n}x=y$, so $T^qx=z$. This means that

$$z\in T^q\Delta(C_M C_{M-1}\cdots C_{n+1})\cap T^n\Delta(id_1^{(i)}\cdots d_{n-1}^{(i)})\cap\Delta(d_1\cdots d_p).$$

So $T^qC=\Delta(d_1\cdots d_p)$ and this proves the claim.

Let $D=\pi^{-1}(C)\cap R_0\times\{0\}\times\{0\}$. Then $\widehat{T}^qD=\Delta(d_1\cdots d_p)\times\Delta(b_1\cdots b_q)\times\{i\}\times\{n\}$. So,

$$\Delta(d_1\cdots d_p)\times\Delta(b_1\cdots b_q)\times\{i\}\times\{n\}\in\bigvee_{n=0}^{\infty}\widehat{T}^n\pi^{-1}(\mathcal{B})$$

and this proves the lemma. $\square$

This leads to the following theorem.

Theorem 3.3.1. *The dynamical system $(R, \mathcal{R}, \nu, \widehat{T})$ is a version of the natural extension of the dynamical system $([0,1), \mathcal{B}, \mu, T)$, where $\mu = \nu \circ \pi^{-1}$ is an invariant probability measure of T, equivalent to the Lebesgue measure, whose density function, $h : [0,1) \to [0,1)$, is given by*

$$h(x) = \frac{2}{16-7\beta}[\beta^3\, 1_{[0,1/2\beta^3)}(x) + (2+\beta)\, 1_{[1/2\beta^3,1/2\beta^2)}(x) \tag{3.7}$$
$$+2\beta\, 1_{[1/2\beta^2,1/2\beta)}(x) + \beta^2\, 1_{[1/2\beta,1/2)}(x) + \beta\, 1_{[1/2,\beta/2)}(x) + 1_{[\beta/2,1)}(x)].$$

Proof. The first part of the proof follows from Remark 3.4.3, the properties of π and Lemma 3.3.2. Let $c = \frac{2}{16-7\beta}$. Then we have for the density,

$$\begin{aligned}
h(x) &= c\Bigg[1 + \frac{1}{\beta}1_{[0,1/2\beta^3)}(x) + \frac{1}{\beta^2}1_{[0,1/2\beta^2)(x)} + \frac{1}{\beta}1_{[0,1/2)}(x) + \frac{1}{\beta^2}1_{[0,\beta/2)}(x) \\
&\quad + \frac{2}{\beta^3}\sum_{i=1}^{\infty}\frac{1}{\beta^{3i}}1_{[0,1/2\beta)}(x) + \frac{2}{\beta^4}\sum_{i=1}^{\infty}\frac{1}{\beta^{3i}}1_{[0,1/2)}(x) + \frac{2}{\beta^5}\sum_{i=1}^{\infty}\frac{1}{\beta^{3i}}1_{[0,\beta/2)}(x)\Bigg] \\
&= c\Bigg[1 + \frac{1}{\beta}1_{[0,1/2\beta^3)}(x) + \frac{1}{\beta^2}1_{[0,1/2\beta^2)(x)} + \frac{1}{\beta}1_{[0,1/2\beta)}(x) \\
&\quad + 1_{[0,1/2)}(x) + \frac{1}{\beta}1_{[0,\beta/2)}(x)\Bigg] \\
&= c\,[\beta^3\, 1_{[0,1/2\beta^3)}(x) + (2+\beta)\, 1_{[1/2\beta^3,1/2\beta^2)}(x) + 2\beta\, 1_{[1/2\beta^2,1/2\beta)}(x) \\
&\quad + \beta^2\, 1_{[1/2\beta,1/2)}(x) + \beta\, 1_{[1/2,\beta/2)}(x) + 1_{[\beta/2,1)}(x)].
\end{aligned}$$

This gives the theorem. □

3.3.2 *Towering the orbits*

For the second version of the natural extension, we will define a transformation on a certain subset of $[0,1) \times [0,\beta)$, using the transformation $\widehat{T}$, defined in the previous section. Define for $n \geq 1$ the following intervals:

$$F_{(1,n)} = \left[\frac{1}{\beta^2} + \frac{1}{\beta^2}\sum_{k=1}^{n-1}\frac{1}{\beta^k}, \frac{1}{\beta^2} + \frac{1}{\beta^2}\sum_{k=1}^{n}\frac{1}{\beta^k}\right)$$

and

$$F_{(3/2,n)} = \left[1 + \frac{1}{\beta^2}\sum_{k=1}^{n-1}\frac{1}{\beta^k}, 1 + \frac{1}{\beta^2}\sum_{k=1}^{n}\frac{1}{\beta^k}\right),$$

where $\sum_{k=1}^{0}\frac{1}{\beta^k} = 0$. Let $F_0 = \left[0, \frac{1}{\beta^2}\right)$. Notice that all of these rectangles are disjoint and that $\bigcup_{n=1}^{\infty} F_{(1,n)} = \left[\frac{1}{\beta^2}, 1\right)$ and $\bigcup_{n=1}^{\infty} F_{(3/2,n)} = [1, \beta)$, so that these intervals together with F_0 form a partition of $[0,\beta)$. Now define the subset $F \subseteq [0,1) \times [0,\beta)$ by

$$F = ([0,1) \times F_0) \cup \bigcup_{n=1}^{\infty}\Big(([0, T^{n-1}(1/2)) \times F_{(1,n)}) \cup ([0, T^{n-1}(1/2\beta^3)) \times F_{(3/2,n)})\Big)$$

and let the function $\phi : F \to R$ be given by

$$\phi(x,y) = \begin{cases} (x, \beta^2(y - \frac{1}{\beta^2} - \frac{1}{\beta^2}\sum_{k=0}^{n-1}\frac{1}{\beta^k}), 1, n), & \text{if } y \in F_{(1,n)}, \\ (x, \beta^2(y - 1 - \frac{1}{\beta^2}\sum_{k=0}^{n-1}\frac{1}{\beta^k}), 3/2, n), & \text{if } y \in F_{(3/2,n)}, \\ (x, \beta^2 y, 0, 0), & \text{if } y \in F_0. \end{cases}$$

So, ϕ maps F_0 to R_0 and for all $n \geq 1$, $i \in \{1, \frac{3}{2}\}$, ϕ maps $F_{(i,n)}$ to $R_{(i,n)}$. Clearly, ϕ is a measurable bijection. Define the transformation $\widetilde{T} : F \to F$, by $\widetilde{T}(x,y) = (\phi^{-1} \circ \widehat{T} \circ \phi)(x,y)$. It is straightforward to check that $\widetilde{T}$ is invertible. In Figure 3.5 we see this transformation.

Let $\mathcal{F}$ be the collection of Borel sets on F. If λ^2 is the 2-dimensional Lebesgue measure, then $\lambda^2(F) = \bar{\lambda}(R)/\beta^2$. Recall that ν is the normalized Lebesgue measure on $(R, \mathcal{R})$. Define a measure $\tilde{\nu}$ on $(F, \mathcal{F})$ by setting $\tilde{\nu}(E) = (\nu \circ \phi)(E)$, for all $E \in \mathcal{F}$. Then $\tilde{\nu}$ is the normalized 2-dimensional Lebesgue measure on $(F, \mathcal{F})$ and the projection of $\tilde{\nu}$ on the first coordinate gives μ again. Also, ϕ is measure preserving and the systems $(R, \mathcal{R}, \mu, \widehat{T})$ and $(F, \mathcal{F}, \tilde{\nu}, \widetilde{T})$ are isomorphic. The following lemma is now enough to show that $(F, \mathcal{F}, \tilde{\nu}, \widetilde{T})$ is a version of the natural extension of $([0,1), \mathcal{B}, \mu, T)$.

Proposition 3.3.1. *The σ-algebras $\mathcal{F}$ and $\bigvee_{n=0}^{\infty} \widetilde{T}^n \pi^{-1}(\mathcal{B})$ are equal.*

Proof. It is easy to see that $\bigvee_{n=0}^{\infty} \widetilde{T}^n \pi^{-1}(\mathcal{B}) \subseteq \mathcal{F}$. For the other inclusion, note that the restriction of the direct products of full fundamental intervals contained in

$$([0,1) \times F_0) \cup \bigcup_{n=1}^{\infty} ([0, T^{n-1}1) \times F_{(1,n)}),$$

generate the restriction of $\mathcal{F}$ to this set. If $\Delta(b_1 \cdots b_n) \in \Delta^{(n)}$ is full in $[0, \frac{1}{\beta})$, then the set $1 + \Delta(b_1 \cdots b_n)$ is a subset of $[1, \beta)$. So, take the direct products of full fundamental intervals in $[0, \beta)$ and sets of the form $1 + \Delta(b_1 \cdots b_n)$ and consider the restriction of these direct products to the set $\bigcup_{n=1}^{\infty} ([0, T^{n-1}(\frac{1}{2\beta^3})) \times F_{(3/2,n)})$. Then these restrictions generate the restriction of the σ-algebra $\mathcal{F}$ to this set. Since $\widetilde{T}$ is isomorphic to $\widehat{T}$, the fact that

$$\mathcal{F} \subseteq \bigvee_{n=0}^{\infty} \widetilde{T}^n \pi^{-1}(\mathcal{B})$$

now can be proven in a way similar to the proof of Lemma 3.3.2. □

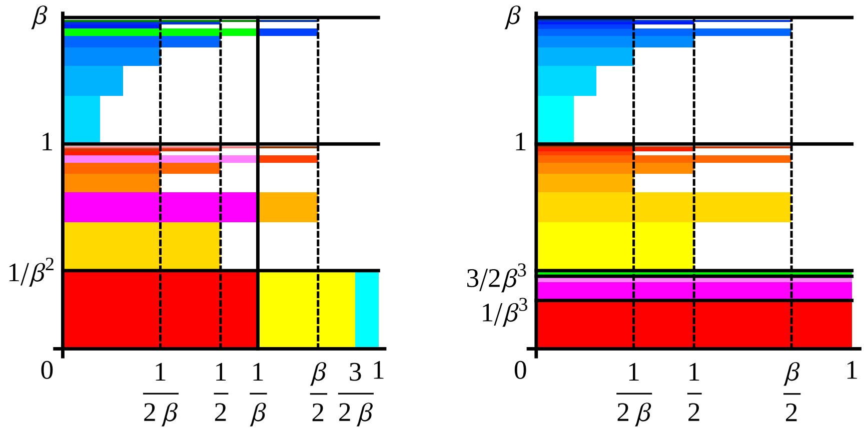


Figure 3.5 The transformation $\widetilde{T}$ maps the regions on the left to the regions on the right with the same color.

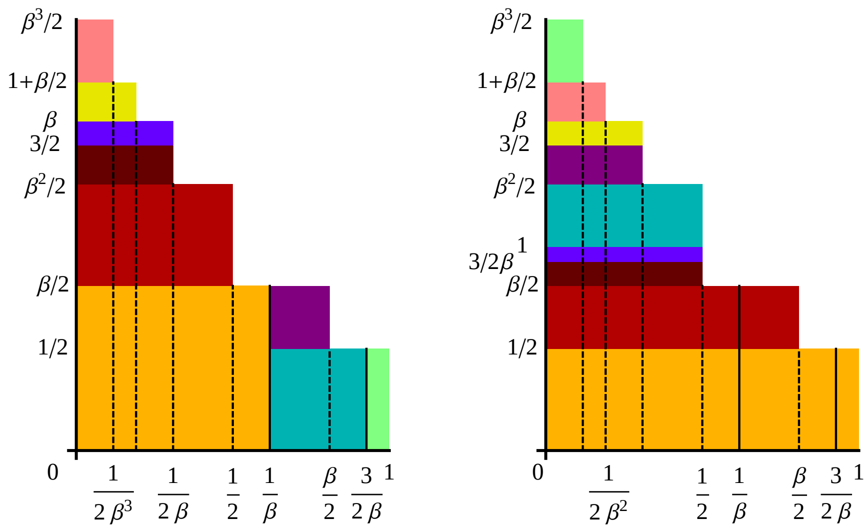


Figure 3.6 The transformation $\overline{T}$ maps the regions on the left to the regions on the right with the same color.

3.3.3 *A special version*

For the last version, we first define the appropriate region in $\mathbb{R}^2$, given by the sets E_1 to E_{12}, as follows.

$$\begin{aligned}
E_1 &= \left[0, \frac{1}{\beta}\right) \times \left[0, \frac{1}{2}\right), & E_2 &= \left[\frac{1}{\beta}, \frac{3}{2\beta}\right) \times \left[0, \frac{1}{2}\right), \\
E_3 &= \left[\frac{3}{2\beta}, 1\right) \times \left[0, \frac{1}{2}\right), & E_4 &= \left[0, \frac{1}{\beta}\right) \times \left[\frac{1}{2}, \frac{\beta}{2}\right), \\
E_5 &= \left[\frac{1}{\beta}, \frac{\beta}{2}\right) \times \left[\frac{1}{2}, \frac{\beta}{2}\right), & E_6 &= \left[0, \frac{1}{2}\right) \times \left[\frac{\beta}{2}, \frac{3}{2\beta}\right), \\
E_7 &= \left[0, \frac{1}{2}\right) \times \left[\frac{3}{2\beta}, 1\right), & E_8 &= \left[0, \frac{1}{2}\right) \times \left[1, \frac{\beta^2}{2}\right), \\
E_9 &= \left[0, \frac{1}{2\beta}\right) \times \left[\frac{\beta^2}{2}, \frac{3}{2}\right), & E_{10} &= \left[0, \frac{1}{2\beta}\right) \times \left[\frac{3}{2}, \beta\right), \\
E_{11} &= \left[0, \frac{1}{2\beta^2}\right) \times \left[\beta, 1 + \frac{\beta}{2}\right), & E_{12} &= \left[0, \frac{1}{2\beta^3}\right) \times \left[1 + \frac{\beta}{2}, \beta + \frac{1}{2}\right).
\end{aligned}$$

Let $E = \bigcup_{i=1}^{12} E_i$. On E, let $\overline{\mathcal{L}}$ be the Lebesgue σ-algebra and let $\overline{\lambda}$ be the normalized Lebesgue measure on $(E, \overline{\mathcal{L}})$. Notice that the projection of $\overline{\lambda}$ on the first coordinate gives the measure μ with the density from (3.7) again. Take $(x, y) \in E$ and suppose the greedy expansion of x is given by $x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$. Define the sequence of numbers $(d_n(x, y))_{n \geq 1}$ as follows. First, let

$$d_1(x, y) = \begin{cases} b_1(x) + \frac{1}{\beta}, & \text{if } (x, y) \in E_3, \\ b_1(x) + 1, & \text{if } (x, y) \in E_{11} \cup E_{12}, \\ b_1(x), & \text{otherwise.} \end{cases}$$

Define the transformation $\overline{T} : E \to E$ by

$$\overline{T}(x, y) = \left(Tx, \frac{y}{\beta} + d_1(x, y)\right).$$

For $n \geq 1$, set $d_n(x, y) = d_1(\overline{T}^{n-1}(x, y))$. It is easy to show that $\overline{T}$ is invertible and measure preserving with respect to $\overline{\lambda}$. For $n \geq 1$ we have

$$\overline{T}^n(x, y) = \left(T^n x, d_n(x, y) + \frac{d_{n-1}(x, y)}{\beta} + \cdots + \frac{d_1(x, y)}{\beta^{n-1}} + \frac{y}{\beta^n}\right).$$

A picture of the transformation is given in Figure 3.6.

It is easy to see that the transformation $\overline{T}$ satisfies (ne1)-(ne4) with factor map π. We can use Theorem 3.1.1 to show that we also have (ne5). Before we can do this, we would like to have an expression for the inverse transformation $\overline{T}^{-1}$. Therefore, we need to define the digit sequence on the elements y and a transformation that generates these digits. Define the numbers $d_1^*(y)$, for

$y \in [0, \beta^3/2)$ by

$$d_1^*(y) = \begin{cases} 0, & \text{if } y \in [0,1), \\ \beta, & \text{if } y \in [1, 3/2), \\ \beta/2, & \text{if } y \in [3/2, 1+\beta/2), \\ 3\beta/2 + 1/2, & \text{if } y \in [1+\beta/2, \beta^3/2). \end{cases}$$

Define the transformation U from the interval $[0, \beta^3/2)$ to itself by $Uy = \beta y - d_1^*(y)$. Also define the numbers

$$b_1^*(y) = \begin{cases} 0, & \text{if } y \in [0,1) \cup [3/2, 1+\beta/2), \\ 1, & \text{if } y \in [1, 3/2), \\ 3/2, & \text{if } y \in [1+\beta/2, \beta^3/2). \end{cases}$$

These numbers b_1^* are used to determine what happens in the first coordinate under the inverse of $\overline{T}$, if we know what $d_1^*(y)$ is. The numbers b_1^* correspond to the numbers d_1^* in the following way.

$$\begin{aligned} d_1^* = 0 \quad &\Rightarrow \quad b_1^* = 0, \\ d_1^* = \beta \quad &\Rightarrow \quad b_1^* = 1, \\ d_1^* = \frac{\beta}{2} \quad &\Rightarrow \quad b_1^* = 0, \\ d_1^* = \frac{3\beta+1}{2} \quad &\Rightarrow \quad b_1^* = \frac{3}{2}. \end{aligned}$$

Let the digit sequence $(d_n^*(y))_{n\geq 1}$ from U be given by $d_n^*(y) = d_1^*(U^{n-1}y)$. Define the sequence $(b_n^*(y))_{n\geq 1}$ similarly. Then it is easily seen that

$$\overline{T}^{-1}(x,y) = \left(\frac{x + b_1^*(y)}{\beta}, Uy\right)$$

and

$$\overline{T}^{-n}(x,y) = \left(\frac{b_n^*(y)}{\beta} + \cdots + \frac{b_1^*(y)}{\beta^n} + \frac{x}{\beta^n}, U^n y\right).$$

We see the transformation U in Figure 3.7. Every element from $[0, \beta^3/2)$ has a digit sequence $(d_n^*(y))_{n\geq 1}$. Moreover, each element has only one digit sequence given by U, i.e., $y \neq y'$ if and only if $(d_n^*(y))_{n\geq 1} \neq (d_n^*(y'))_{n\geq 1}$. There are two digits d_1^* that correspond to the same digit b_1^*. This could imply that the digit sequences $(b_n^*)_{n\geq 1}$ are not unique. Fortunately, this is not true. For a sequence $(b_n^*)_{n\geq 1}$, there is still a way to know if a 0 comes from a 0 or from a $\beta/2$. Note that if for some n, $d_n^*(y) = \beta/2$, then

$$d_n^*(y) d_{n+1}^*(y) = \frac{\beta}{2}\,\frac{3\beta+1}{2} \quad \text{or} \quad d_n^*(y) d_{n+1}^*(y) d_{n+2}^*(y) = \frac{\beta}{2}\,\frac{\beta}{2}\,\frac{3\beta+1}{2}$$

and hence,

$$b_n^*(y) b_{n+1}^*(y) = 0\,\frac{3}{2} \quad \text{or} \quad b_n^*(y) b_{n+1}^*(y) b_{n+2}^*(y) = 00\,\frac{3}{2}.$$

On the other hand, if $d_n^*(y) = 0$, then we can never have $d_{n+1}^*(y) = (3\beta+1)/2$ or $d_{n+2}^*(y) = (3\beta+1)/2$. So, we have $b_n^*(y) = 0$ and $U^{n-1}y \in [3/2, 1+\beta/2)$ if and only if $b_{n+1}^*(y) = 3/2$ or $b_{n+1}^*(y) b_{n+2}^*(y) = 0\frac{3}{2}$. This implies that if two sequences $(d_n^*(y))_{n\geq 1}$ and $(d_n^*(y'))_{n\geq 1}$ are not equal, then the corresponding sequences $(b_n^*(y))_{n\geq 1}$ and $(b_n^*(y'))_{n\geq 1}$ are not equal as well. We need this in the proof of the next proposition, which gives (ne5).

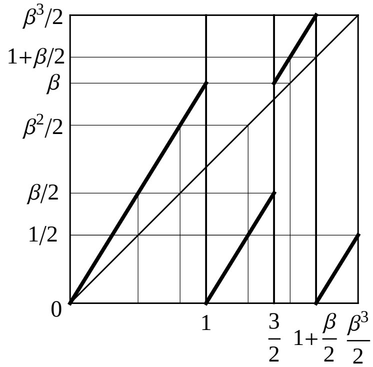


Figure 3.7 The transformation U.

Proposition 3.3.2. *For the dynamical system* $(E, \overline{\mathcal{L}}, \overline{\lambda}, \overline{T})$, *we have that*

$$\overline{\mathcal{L}} = \bigvee_{n=0}^{\infty} \overline{T}^n \pi^{-1}(\mathcal{L}).$$

Proof. Since $\overline{T}$ maps rectangles to rectangles, it is clear that

$$\bigvee_{n=0}^{\infty} \overline{T}^n \pi^{-1}(\mathcal{L}) \subseteq \overline{\mathcal{L}}.$$

For the other inclusion, let (x_1, y_1) and (x_2, y_2) be two different points from E. If $x_1 \neq x_2$, then there are two intervals $B_1, B_2 \subseteq [0, 1)$, such that $x_1 \in B_1$, $x_2 \in B_2$ and $B_1 \cap B_2 = \emptyset$. Hence,

$$(x_1, y_1) \in \pi^{-1} B_1 \in \bigvee_{n=0}^{\infty} \overline{T}^n \pi^{-1}(\mathcal{L}), \text{and } (x_2, y_2) \in \pi^{-1} B_2 \in \bigvee_{n=0}^{\infty} \overline{T}^n \pi^{-1}(\mathcal{L})$$

and $\pi^{-1} B_1 \cap \pi^{-1} B_2 = \emptyset$. Now suppose that $x_1 = x_2 = x$ and $y_1 \neq y_2$. Then the sequences $(d_n^*(y_1))_{n \geq 1}$ and $(d_n^*(y_2))_{n \geq 1}$ are not equal and thus the same holds for the sequences $(b_n^*(y_1))_{n \geq 1}$ and $(b_n^*(y_2))_{n \geq 1}$. Let N be the first index such that

$$b_1^*(y_1) \cdots b_N^*(y_1) \neq b_1^*(y_2) \cdots b_N^*(y_2).$$

Consider the points

$$x_1^{(N)} = \frac{b_N^*(y_1)}{\beta} + \cdots + \frac{b_1^*(y_1)}{\beta^N} + \frac{x}{\beta^N} \quad \text{and} \quad x_2^{(N)} = \frac{b_N^*(y_2)}{\beta} + \cdots + \frac{b_1^*(y_2)}{\beta^N} + \frac{x}{\beta^N}.$$

By the choice of N, $x_1^{(N)} \neq x_2^{(N)}$. This means that there are two disjoint intervals $B_1, B_2 \in \mathcal{L}$, such that $x_1^{(N)} \in B_1$ and $x_2^{(N)} \in B_2$. Also,

$$(x, y_1) \in \overline{T}^N \pi^{-1}(B_1) \quad \text{and} \quad (x, y_2) \in \overline{T}^N \pi^{-1}(B_2).$$

Moreover, since B_1 and B_2 are disjoint, by the injectivity of $\overline{T}$,

$$T^N \pi^{-1}(B_1) \cap T^N \pi^{-1}(B_2) = \emptyset.$$

By Theorem 3.1.1, this implies the proposition. $\square$

So by Proposition 3.3.2 also the system $(E, \overline{\mathcal{L}}, \overline{\lambda}, \overline{T})$ is a version of the natural extension of the system $([0,1), \mathcal{L}, \mu, T)$.

3.4 More on fundamental intervals

We now return to the original problem. We want to define a version of the natural extension of the greedy β-transformation with allowable digit set $A = \{0, 1, u\}$, such that β satisfies (3.1). Recall that condition (3.1) implies that the support of the ACIM of the transformation is the interval $[0, 1)$. The way we define this version is very similar to what is done in Section 3.3.1. A very important feature of the example, studied in the previous section, is that after one iteration the endpoints of the transformation always stay in the region $\Delta(0) \cup \Delta(1)$. In general this does not hold and in what follows, we will see that this complicates matters. First we look at the fundamental intervals again.

As before, for $n \geq 1$, let D_n be the collection of all non-full fundamental intervals of rank n that are not contained in any full fundamental interval of lower rank. Let $\kappa(n)$ be the number of elements in D_n. So $\kappa(1) = 2$, since $D_1 = \{\Delta(1), \Delta(u)\}$ and for all $n \geq 1$, $\kappa(n) \leq 2^n$. The version of the natural extension that we will define, uses all the elements of D_n for all $n \geq 1$. To make sure that the total measure of the underlying space of this version is finite, we need upper bounds for the values of $\kappa(n)$. To obtain these, we will first describe the structure of the elements of D_n. Let

$$
\begin{aligned}
u - 1 &= \sum_{k=1}^{\infty} \frac{d_k^{(1)}}{\beta^k}, \\
\beta - u &= \sum_{k=1}^{\infty} \frac{d_k^{(u)}}{\beta^k},
\end{aligned}
$$

be the greedy β-expansions with digits in A of the points $u-1$ and $\beta - u$. The number $\beta - u$ would be the image of 1 under T if T were not restricted to the interval $[0, 1)$. The values of the numbers $\kappa(n)$ depend on the orbits of the points $\beta - u$ and $u - 1$ under T and whether or not $T^k(u-1)$ and $T^k(\beta - u)$ are elements of $\Delta(u)$ for $0 \leq k < n$. To see this, note that for any $\Delta(b_1 \cdots b_n) \in D_n$ one has $b_1 \in \{1, u\}$. We claim that the set $T^n\Delta(b_1 \cdots b_n)$ has the form

$$[0, T^k(u-1)) \quad \text{or} \quad [0, T^k(\beta - u))$$

for some $0 \leq k < n$. This can be shown by induction. For $n = 1$, we have $T\Delta(1) = [0, u-1)$ and $T\Delta(u) = [0, \beta - u)$. Now, suppose $T^n\Delta(b_1 \cdots b_n) = [0, T^k(u-1))$ for example.

If $\lambda([0, T^k(u-1)) \cap \Delta(u)) = 0$, then $\Delta(b_1 \cdots b_n)$ contains exactly one element of D_{n+1}. This is $\Delta(b_1 \cdots b_n 0)$ in case $\lambda([0, T^k(u-1)) \cap \Delta(1)) = 0$ and $\Delta(b_1 \cdots b_n 1)$ in case $\lambda([0, T^k(u-1)) \cap \Delta(1)) > 0$. Furthermore, in the first case,

$$T^{n+1}\Delta(b_1 \cdots b_n 0) = [0, T^{k+1}(u-1))$$

and also in the second case,

$$T^{n+1}\Delta(b_1 \cdots b_n 1) = [0, T^{k+1}(u-1)).$$

On the other hand, if $\lambda([0, T^k(u-1)) \cap \Delta(u)) > 0$, then $\Delta(b_1 \cdots b_n)$ contains exactly two elements of D_{n+1}, namely the sets $\Delta(b_1 \cdots b_n 1)$ and $\Delta(b_1 \cdots b_n u)$. Now,

$$T^{n+1}\Delta(b_1 \cdots b_n 1) = [0, u-1)$$

and

$$T^{n+1}\Delta(b_1 \cdots b_n u) = [0, T^{k+1}(u-1)).$$

Similar arguments hold in case $T^n\Delta(b_1 \cdots b_n) = [0, T^k(\beta - u))$, except that $T^k(u-1)$ is replaced by $T^k(\beta - u)$.

For $n \geq 1$, let $\bar{\kappa}(n)$ be the number of elements from D_n that contain two elements from D_{n+1}. Then clearly for all $n \geq 1$,

$$\kappa(n+1) = \kappa(n) + \bar{\kappa}(n). \tag{3.8}$$

From the above we see that in order to get an upper bound on $\kappa(n)$, we need to study the orbits of $u-1$ and $\beta - u$. The following three lemmas say something about the first few elements of the orbits of these points. These lemmas are needed to guarantee that the total measure of the underlying space of the natural extension will be finite.

Lemma 3.4.1. *If* $1 < \beta \leq 2$ *and* $u < \beta$, *then* $u - 1 \notin \Delta(u)$.

Proof. Since $\beta \leq 2$, we have $1 - \frac{1}{\beta} \leq \frac{1}{\beta}$. Thus $u\left(1 - \frac{1}{\beta}\right) \leq \frac{u}{\beta} < 1$ and hence $u - 1 < \frac{u}{\beta}$. This proves the lemma. □

As said before, D_n only contains fundamental intervals of which the first digit is either 1 or u. Let $\kappa_1(n)$ denote the number of elements $\Delta(b_1 \cdots b_n)$ in D_n such that $b_1 = 1$ and $\kappa_2(n)$ the number of elements in D_n that have u as their first digit. Then, of course, for all $n \geq 1$,

$$\kappa(n) = \kappa_1(n) + \kappa_2(n).$$

Let $(F(n))_{n \geq 0}$ denote the Fibonacci sequence, i.e., let $F(0) = 0$, $F(1) = 1$ and for $n \geq 2$, let $F(n) = F(n-1) + F(n-2)$. Lemma 3.4.1 implies that for $1 < \beta \leq 2$, the number of elements of D_{n+1} is not larger than the number that we would get if the only elements of D_n that do not contain two elements from D_{n+1} are the elements $\Delta(b_1 \cdots b_n)$ for which $T^n\Delta(b_1 \cdots b_n) = [0, u-1)$. In this maximal situation we would have $\kappa_1(1) = \kappa_1(2) = 1$ and for $n \geq 3$,

$$\kappa_1(n) = \kappa_1(n-1) + \kappa_1(n-2).$$

For κ_2 we would have that $\kappa_2(n) = \kappa_1(n+1)$. This means that under the conditions from Lemma 3.4.1, we have for all $n \geq 1$ that $\kappa_1(n) \leq F(n)$ and

$$\kappa(n) = \kappa_1(n) + \kappa_2(n) \leq F(n) + F(n+1) = F(n+2). \tag{3.9}$$

Let $G = \frac{1+\sqrt{5}}{2}$ be the golden mean, i.e., the positive solution of the equation $x^2 - x - 1 = 0$.

Lemma 3.4.2. *Let* $1 < \beta \leq G$ *and* $u < \beta$. *Then* $u - 1, \beta - u \in \Delta(0)$.

Proof. Since $\beta \le G$, we have $u < \beta \le 1 + \frac{1}{\beta}$. Thus $u - 1 < \frac{1}{\beta}$ and hence $u - 1 \in \Delta(0)$. On the other hand, since $\beta - \frac{1}{\beta} \le 1$, we have $\beta - u < \frac{1}{\beta}$ and thus $\beta - u \in \Delta(0)$. □

Remark 3.4.1. This lemma implies that for $1 < \beta \le G$ and for digit sets satisfying condition (3.1), we have $\kappa(2) = 2$. The largest amount of elements for D_n would be obtained if $\lambda([0, T^k(u-1)) \cap \Delta(u)) > 0$ and $\lambda([0, T^k(\beta - u)) \cap \Delta(u)) > 0$ for all odd values of k and thus $T^k(u-1), T^k(\beta-u) \in \Delta(0)$ for all even values of k. In this case,

$$\bar{\kappa}(n) = \begin{cases} \kappa(n), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

This would imply that for $n \ge 1$, $\kappa(2n-1) = \kappa(2n) = 2^n$. So, if $1 < \beta \le G$, we have that for all $n \ge 1$, $\kappa(n) \le 2^{\lfloor n/2 \rfloor + 1}$.

Lemma 3.4.3. *Let $m \ge 2$ and $u < \beta$. If $1 < \beta \le 2^{\frac{1}{m}}$, then*

$$T^k(u-1),\ T^k(\beta - u) \in \Delta(0)$$

for all $0 \le k \le m - 1$.

Proof. The proof is by induction on m. Note that from Lemma 3.4.2 we know that $u - 1$, $\beta - u \in \Delta(0)$. Assume first that $m = 2$ and thus $\beta \le \sqrt{2} = 2^{\frac{1}{2}}$. Then

$$T(u-1) = \beta(u-1) \in \Delta(0) \quad \Leftrightarrow \quad \beta(u-1) < \frac{1}{\beta} \quad \Leftrightarrow \quad \frac{u}{\beta} < \frac{1}{\beta} + \frac{1}{\beta^3}.$$

If $\beta \le \sqrt{2}$, then $\frac{1}{\beta} + \frac{1}{\beta^3} > 1$ and since $u < \beta$, $T(u-1) \in \Delta(0)$. On the other hand, since $\beta - u \in \Delta(0)$, we have

$$T(\beta - u) = \beta(\beta - u) \in \Delta(0) \quad \Leftrightarrow \quad \beta^2 - \beta u < \frac{1}{\beta} \quad \Leftrightarrow \quad \beta - \frac{1}{\beta^2} < u.$$

If $\beta \le \sqrt{2}$, then $\beta - \frac{1}{\beta^2} < 1$ and thus $T(\beta - u) \in \Delta(0)$.

Now, assume that the result is true for some $k \ge 2$. Let $m = k + 1$ and $\beta \le 2^{\frac{1}{k+1}}$. Then certainly $\beta \le 2^{\frac{1}{k}}$, so by induction $T^j(u-1)$, $T^j(\beta - u) \in \Delta(0)$ for all $0 \le j \le k - 1$. We only need to show that $T^k(u-1)$, $T^k(\beta - u) \in \Delta(0)$. First consider $T^k(u-1) = \beta^k(u-1)$. We have

$$T^k(u-1) \in \Delta(0) \quad \Leftrightarrow \quad \beta^k(u-1) < \frac{1}{\beta} \quad \Leftrightarrow \quad \frac{u}{\beta} < \frac{1}{\beta} + \frac{1}{\beta^{k+2}}.$$

Since $\beta \le 2^{\frac{1}{k+1}}$, then

$$\frac{1}{\beta} + \frac{1}{\beta^{k+2}} \ge \frac{3}{2 \cdot 2^{\frac{1}{k+1}}} \ge \frac{3}{2\sqrt{2}} > 1.$$

Thus $T^k(u-1) \in \Delta(0)$. We now consider $T^k(\beta - u) = \beta^k(\beta - u)$ and see that

$$T^k(\beta - u) \in \Delta(0) \quad \Leftrightarrow \quad \beta^{k+1} - \beta^k u < \frac{1}{\beta} \quad \Leftrightarrow \quad \beta - \frac{1}{\beta^{k+1}} < u.$$

Since $\beta \le 2^{\frac{1}{k+1}}$, then

$$\beta - \frac{1}{\beta^{k+1}} \le 2^{\frac{1}{k+1}} - \frac{1}{2} \le \sqrt{2} - \frac{1}{2} < 1.$$

Thus $T^k(\beta - u) \in \Delta(0)$ and this proves the lemma. □

Remark 3.4.2. Suppose $2^{1/(m+1)} < \beta \leq 2^{1/m}$ and $u < \beta$. Lemma 3.4.3 implies that $\kappa(i) = 2$ for $1 \leq i \leq m+1$. By the same reasoning as in Remark 3.4.1, $\kappa(n)$ would obtain the largest possible value if

$$\lambda([0, T^i(u-1)) \cap \Delta(u)) > 0 \quad \text{and} \quad \lambda([0, T^i(\beta - u)) \cap \Delta(u)) > 0$$

for all $i = \ell m + (\ell - 1)$, $\ell \geq 1$, and $T^i(u-1), T^i(\beta - u) \in \Delta(0)$ for all other values of i. This would imply

$$\bar{\kappa}(n) = \begin{cases} \kappa(n), & \text{if } n = \ell m + (\ell - 1) \text{ for some } \ell \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, to get the maximal number of elements for D_n, we would have that if

$$(\ell - 1)m + \ell \leq n \leq \ell m + \ell$$

for some ℓ, then $\kappa(n) = 2^\ell$. So, in general we have $\ell \leq \frac{n+m}{m+1}$ and thus $\kappa(n) \leq 2 \cdot 2^{\frac{n}{m+1}}$.

For all $n \geq 1$, let B_n be the union of all full fundamental intervals of rank n, that are not a subset of any full fundamental interval of lower rank. From the next lemma it follows that the full fundamental intervals generate the Borel σ-algebra on $[0, 1)$. Note that we have already proved this in Lemma 2.4.1 for $2 < \beta < 3$.

Lemma 3.4.4.

$$\lambda\left(\bigcup_{n \geq 1} B_n\right) = \sum_{n \geq 1} \lambda(B_n) = \lambda([0, 1)) = 1.$$

Proof. Note that all of the sets B_n are disjoint. By (3.3) we have for each $n \geq 1$, that

$$0 \leq \lambda\left([0, 1) \setminus \bigcup_{i=1}^{n} B_i\right) \leq \frac{\kappa(n)}{\beta^n},$$

so it is enough to prove that $\lim_{n \to \infty} \frac{\kappa(n)}{\beta^n} = 0$. If $2 < \beta < 3$, then since $\kappa(n) \leq 2^n$, we immediately have the result. For $1 < \beta \leq G$, it follows from Remark 3.4.1 and Remark 3.4.2. Now, suppose $G < \beta \leq 2$. Then by (3.9), we have that $\kappa(n) \leq F(n+2)$, where $F(n+2)$ is the $(n+2)$-th element of the Fibonacci sequence with $F(0) = 0$ and $F(1) = 1$. For the elements of this sequence, there is a closed formula which gives

$$F(n) = \frac{G^n - (1-G)^n}{\sqrt{5}}. \tag{3.10}$$

So

$$\frac{\kappa(n)}{\beta^n} \leq \frac{1}{\sqrt{5}} \left[G^2 \left(\frac{G}{\beta}\right)^n - (1-G)^2 \left(\frac{1-G}{\beta}\right)^n\right].$$

Since $G < \beta \leq 2$, also in this case $\lim_{n \to \infty} \frac{\kappa(n)}{\beta^n} = 0$ and this proves the lemma. □

Remark 3.4.3. The fact that $\Delta(0)$ is a full fundamental interval of rank 1 allows us to construct full fundamental intervals of arbitrary small Lebesgue measure. This, together with the previous lemma, guarantees that we can write each subinterval of $[0,1)$ as a countable union of full fundamental intervals. Thus, the full fundamental intervals generate the Borel σ-algebra on $[0,1)$.

Notice that for the cases illustrated by Figure 3.2(b) and 3.2(c), we can define the numbers $\kappa(n)$ in a similar way. The only differences are that the support of the ACIM is given by the interval $[0, u-1)$ and that $\Delta(1)$ is the only full fundamental interval of rank 1. In that sense, $\Delta(1)$ plays the role of $\Delta(0)$ above. Since in these cases we have $2 < \beta < 3$ and since $\kappa(n) \leq 2^n$ for all $n \geq 1$, we can prove a lemma similar to Lemma 3.4.4, i.e., we can prove that the full fundamental intervals generate the Borel σ-algebra on the support of the ACIM.

3.5 Back from the future

The transformation of a natural extension is invertible. Each time we apply a greedy β-transformation T on an x, we loose the first digit of the expansion of that x and since T is not injective, most of the times Tx has several possible inverse images or 'pasts'. To construct a natural extension for T, we must find a way to remember these pasts. In Section 3.3.1 this was done by introducing the space of rectangles R. We will use a similar strategy for the general three digit case.

3.5.1 *The natural extension in general*

For the natural extension, we will define a space R, using all the elements of D_n. Such a space is shown in Figure 3.10 in Section 3.5.2. For $n \geq 1$, define the collections of rectangles

$$R_n = \left\{ T^n \Delta(b_1 \cdots b_n) \times \left[0, \frac{1}{\beta^n}\right) \,\middle|\, \Delta(b_1 \cdots b_n) \in D_n \right\}.$$

So, to each element of D_n, there corresponds an element of R_n and thus the number of elements in R_n is given by $\kappa(n)$. We enumerate the elements of R_n and write $R_n = \{R_{(n,j)} \,|\, 1 \leq j \leq \kappa(n)\}$. Thus, for each $\Delta(b_1 \cdots b_n) \in D_n$ there exists a unique $1 \leq j \leq \kappa(n)$ such that $T^n \Delta(b_1 \cdots b_n) \times [0, \frac{1}{\beta^n}) = R_{(n,j)}$. Let $R_0 = [0,1) \times [0,1)$ and let R be the disjoint union of all these sets, i.e.,

$$R = R_0 \times \{0\} \times \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\kappa(n)} R_{(n,j)} \times \{n\} \times \{j\}.$$

Note that the meaning of the extra two coordinates n and j differs from how the extra two coordinates were defined in Section 3.3.1. This is a consequence of the fact that in general we cannot explicitly give the elements from D_n. The σ-algebra $\mathcal{R}$ on R is the disjoint union of the Borel σ-algebras on all the rectangles $R_{(n,k)}$ and R_0. Let λ_R be the measure on R, given by the two dimensional Lebesgue measure on each of these rectangles. Define the probability measure ν on R by setting $\nu(E) = \frac{1}{\lambda_R(R)}\lambda_R(E)$ for all $E \in \mathcal{R}$. The next lemma says that this measure is well defined and finite.

Lemma 3.5.1. $\lambda_R(R) < \infty$.

Proof. Set $\kappa(0) = 1$. Then,

$$\lambda_R(R) \leq \sum_{n=0}^{\infty} \frac{\kappa(n)}{\beta^n}.$$

Using the same arguments as in the proof of Lemma 3.4.4, we can show that the sum on the right hand side converges for all $1 < \beta < 3$. □

Remark 3.5.1. (i)The result from the previous lemma is what makes the general case more difficult than the example from Section 3.3. For that example, we have $\kappa(n) = 2$ for all n, which follows from the orbits of the points $1/2$ and $1/2\beta^3$. This immediately implies that the space R has finite measure. In order to establish the same for general transformations, we need everything from Section 3.4.
(ii) Consider the natural extension for the GLS($\mathcal{I}$)-transformation from Example 3.1.1. We can define fundamental intervals for this transformation by setting $\Delta = \mathcal{I} \cup D$ and for $n \geq 1$, $\Delta^{(n)} = \bigvee_{k=0}^{n-1} S^{-k}\Delta$. Notice that for the GLS-transformation, all fundamental intervals of all ranks are full. This implies that in GLS($\mathcal{I}$)-expansions all digits can in principle be followed by all other digits. As a consequence, for these transformations the space R would only consist of the rectangle $[0, 1)$.

The transformation $\widehat{T} : R \to R$ is defined piecewise on each rectangle. If $(x, y) \in R_0$, then

$$\widehat{T}(x, y, 0, 0) = \begin{cases} (Tx, \frac{y}{\beta}, 0, 0), & \text{if } x \in \Delta(0), \\ (Tx, \frac{y}{\beta}, 1, 1), & \text{if } x \in \Delta(1), \\ (Tx, \frac{y}{\beta}, 1, 2), & \text{if } x \in \Delta(u). \end{cases}$$

For each $n \geq 1$ and $1 \leq j \leq \kappa(n)$, if

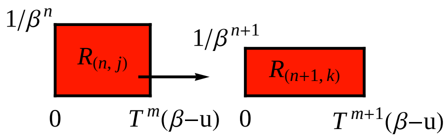
$$R_{(n,j)} = T^n\Delta(b_1 \cdots b_n) \times \left[0, \frac{1}{\beta^n}\right),$$

then $\widehat{T}$ maps this rectangle to the rectangles corresponding to the fundamental intervals contained in $\Delta(b_1 \cdots b_n)$ in the following way. If $\Delta(b_1 \cdots b_n 0)$ is full and $(x, y) \in R_{(n,j)}$ with $x \in \Delta(0)$, then

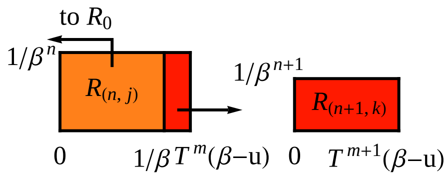
$$\widehat{T}(x, y, n, j) = \left(Tx, \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_n}{\beta^n} + \frac{y}{\beta}, 0, 0\right).$$

If $\Delta(b_1 \cdots b_n b_{n+1}) \in D_{n+1}$ and k is the index of the corresponding set in R_{n+1}, then for $(x, y) \in R_{(n,j)}$ with $x \in \Delta(b_{n+1})$, we set

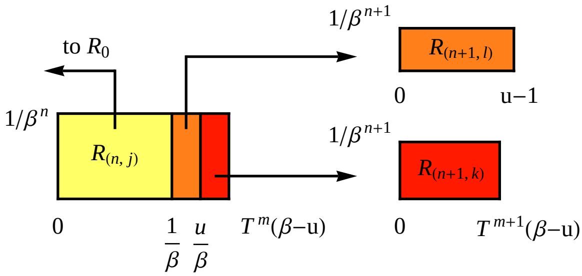
$$\widehat{T}(x, y, n, j) = \left(Tx, \frac{y}{\beta}, n + 1, k\right).$$



(a) $\lambda([0, T^m(\beta-u)) \cap \Delta(1)) = 0$



(b) $\lambda([0, T^m(\beta-u)) \cap \Delta(1)) > 0$ and $\lambda(T^m(\beta-u) \cap \Delta(u)) = 0$



(c) $\lambda([0, T^m(\beta-u)) \cap \Delta(u)) > 0$

Figure 3.8 The arrows indicate the action of $\widehat{T}$ in case $T^n\Delta(b_1\cdots b_n) = [0, T^m(\beta-u))$, for some $m < n$.

In Figure 3.8 we show the different situations in case $T^n\Delta(b_1\cdots b_n) = T^m(\beta-u)$ for some $m < n$. If $T^m\Delta(b_1\cdots b_n) = T^m(u-1)$ for some $m < n$, the pictures look exactly the same with $T^m(u-1)$ in place of $T^m(\beta-u)$.

If a rectangle $R_{(n,j)}$ corresponds to a fundamental interval $\Delta(b_1\cdots b_n)$ such that $\Delta(b_1\cdots b_n 0)$ is non-full, then this is the only fundamental interval contained in $\Delta(b_1\cdots b_n)$. $\widehat{T}$ then maps the rectangle $R_{(n,j)}$ bijectively to the rectangle $R_{(n+1,k)}$, corresponding to $\Delta(b_1\cdots b_n 0)$. Otherwise, $\Delta(b_1\cdots b_n 0)$ is a full fundamental interval contained in $\Delta(b_1\cdots b_n)$ and the rectangle $R_{(n,j)}$ is partly mapped surjectively onto some rectangle $R_{(n+1,k)}$ and partly into R_0. From Lemma 3.4.4 it now follows that $\widehat{T}$ is bijective.

Let $\pi : R \to [0,1)$ be the projection onto the first coordinate. Define the measure μ on $([0,1),\mathcal{B})$ by pulling back the measure ν, i.e., for all measurable sets $E \in \mathcal{B}$, let $\mu(E) = \nu(\pi^{-1}E)$. In order to show that $(R, \mathcal{R}, \nu, \mathcal{T})$ is a version of the natural extension of $([0,1),\mathcal{B},\mu,T)$ with π as the factor map, we will prove (ne1)-(ne5) from Section 3.1. It is clear that π is surjective and measurable and that $T\circ\pi = \pi\circ\widehat{T}$. Since $\widehat{T}$ expands by a factor β in the first coordinate and contracts by a factor β in the second coordinate, it is also clear that $\widehat{T}$ is invariant with respect to the measure ν. Then $\mu = \nu\circ\pi^{-1}$ defines a T-invariant probability measure on $([0,1),\mathcal{B})$, that is equivalent to the Lebesgue measure, and π is a measure preserving map. This shows (ne1) to (ne4). The invertibility of $\widehat{T}$ follows from Remark 3.4.3, so that leaves only (ne5). To prove (ne5) we will introduce the division of full fundamental intervals into full blocks as was done in Section 3.3.1.

Let $\Delta(b_1 \cdots b_n)$ be a full fundamental interval. We can divide the block of digits $b_1 \cdots b_n$ into M subblocks, $C_1, \ldots, C_M$, for some $M \geq 1$, where each subblock corresponds to a full fundamental interval itself. We can make this precise, using the notion of return times to R_0. For points $(x,y) \in R_0$ define the *first return time to* R_0 by

$$r_1(x,y) = \inf\{j \geq 1 \,|\, \widehat{T}^j(x,y,0,0) \in R_0 \times \{0\} \times \{0\}\}$$

and for $k > 1$, let the *k-th return time to* R_0 be given recursively by

$$r_k(x,y) = \inf\{j > r_{k-1}(x,y) \,|\, \widehat{T}^j(x,y,0,0) \in R_0 \times \{0\} \times \{0\}\}.$$

By the Poincaré Recurrence Theorem, we have $r_k(x,y) < \infty$ for almost all $(x,y) \in R_0$. Notice that this notion of return time depends only on x, i.e., for all $(x,y),(x,y') \in R_0$ and all $k \geq 1$, $r_k(x,y) = r_k(x,y')$. So we can write $r_k(x)$ instead of $r_k(x,y)$. If $\Delta(b_1 \cdots b_n) \in \Delta^{(n)}$, then for all $m \leq n$, $\widehat{T}^m$ maps the whole rectangle $\Delta(b_1 \cdots b_n) \times [0,1) \subseteq R_0$ to the same rectangle in R. So, the first several return times to R_0 are equal for all elements in $\Delta(b_1 \cdots b_n)$. Hence, there is an $M \geq 1$ and there are numbers r_k, $1 \leq k \leq M$, such that $r_k = r_k(x)$ for all $x \in \Delta(b_1 \cdots b_n)$ and $r_M = n$. Put $r_0 = 0$. We can also obtain the numbers r_k inductively as follows. Let

$$r_1 = \inf\{j \geq 1 \,|\, T^j \Delta(b_1 \cdots b_j) = [0,1)\}$$

and if $r_1, \ldots, r_{k-1}$ are already known, let

$$r_k = \inf\{j > r_{k-1} \,|\, T^j \Delta(b_{r_{k-1}+1} \cdots b_j) = [0,1)\}.$$

Take for $1 \leq k \leq M$,

$$C_k = b_{r_{k-1}+1} \cdots b_{r_k}. \tag{3.11}$$

Let $|C_k|$ denote the number of digits of the block C_k. These blocks also have the properties (p1)-(p4) from Section 3.3.1. The next lemma is the last step in proving that $(R, \mathcal{R}, \nu, \widehat{T})$ is the natural extension of the space $([0,1), \mathcal{B}, \mu, T)$.

Lemma 3.5.2. *Let* $(R, \mathcal{R}, \nu, \widehat{T})$ *and* $([0,1), \mathcal{B}, \mu, T)$ *be the dynamical systems defined above. Then*

$$\mathcal{R} = \bigvee_{n=0}^{\infty} \widehat{T}^n(\pi^{-1}(\mathcal{B})).$$

Proof. It is clear that

$$\bigvee_{n=0}^{\infty} \widehat{T}^n(\pi^{-1}(\mathcal{B})) \subseteq \mathcal{R}.$$

By Lemma 3.4.4 we know that the direct products of the full fundamental intervals contained in the rectangle R_0 generate the Borel σ-algebra on this rectangle. The same holds for all the rectangles $R_{(n,k)}$. First, let

$$\Delta(d_1 \cdots d_p) \times \Delta(e_1 \cdots e_q)$$

be a generating rectangle in R_0, where $\Delta(d_1 \cdots d_p)$ and $\Delta(e_1 \cdots e_q)$ are full fundamental intervals. For the set $\Delta(e_1 \cdots e_q)$ construct the subblocks $C_1, \ldots, C_M$ as in (3.11). By property (p4) and Lemma 3.2.1, the set

$$\Delta(C_M C_{M-1} \cdots C_1 d_1 \cdots d_p)$$

is a full fundamental interval of rank $p+q$. Then

$$\begin{aligned}&\widehat{T}^q(\pi^{-1}(C_M C_{M-1} \cdots C_1 d_1 \cdots d_p)) \cap (R_0 \times \{0\} \times \{0\})) \\ &\quad = \Delta(d_1 \cdots d_p) \times \Delta(C_1 C_2 \cdots C_M) \times \{0\} \times \{0\}.\end{aligned}$$

So

$$\Delta(d_1 \cdots d_p) \times \Delta(e_1 \cdots e_q) \times \{0\} \times \{0\} \subseteq \bigvee_{n=0}^{\infty} \widehat{T}^n \pi^{-1}(\mathcal{B}).$$

Now, for $n \geq 1$ and $1 \leq j \leq \kappa(n)$, suppose $R_{(n,j)} \in R_n$ corresponds to the fundamental interval $\Delta(b_1 \cdots b_n) \in D_n$. Hence,

$$R_{(n,j)} = T^n \Delta(b_1 \cdots b_n) \times \left[0, \frac{1}{\beta^n}\right).$$

Let $\Delta(d_1 \cdots d_p) \times \Delta(e_1 \cdots e_q)$ be a generating rectangle for the Borel σ-algebra on the rectangle $R_{(n,j)}$. So $\Delta(d_1 \cdots d_p)$ and $\Delta(e_1 \cdots e_q)$ are again full fundamental intervals. Notice that

$$\Delta(e_1 \cdots e_q) \subseteq \Delta(\underbrace{0 \cdots 0}_{n \text{ times}}),$$

which means that $q \geq n$. Also, for all $1 \leq k \leq n$, $e_k = 0$ and thus $r_k = k$. So, if we divide $e_1 \cdots e_q$ into subblocks C_k as before, we get that $C_1 = C_2 = \cdots = C_n = 0$, that $M \geq n$ and that $|C_{n+1}| + \cdots + |C_M| = q - n$. Consider the set

$$C = \Delta(C_M C_{M-1} \cdots C_{n+1} b_1 \cdots b_n d_1 \cdots d_p).$$

In the same way as was done in the proof of the claim in the proof of Lemma 3.3.2, we can show that the set C is a fundamental interval of rank $p+q$ and that $T^q C = \Delta(d_1 \cdots d_p)$. Set $D = \pi^{-1}(C) \cap (R_0 \times \{0\} \times \{0\})$. Then,

$$\widehat{T}^{q-n} D = \Delta(b_1 \cdots b_n d_1 \cdots d_p) \times \Delta(C_{n+1} C_{n+2} \cdots C_M) \times \{0\} \times \{0\}.$$

And after n more steps,

$$\begin{aligned}\widehat{T}^q D &= \Delta(d_1 \cdots d_p) \times \Delta(\underbrace{00 \cdots 0}_{n \text{ times}} C_{n+1} \cdots C_M) \times \{n\} \times \{j\} \\ &= \Delta(d_1 \cdots d_p) \times \Delta(e_1 \cdots e_q) \times \{n\} \times \{j\}.\end{aligned}$$

So,

$$\Delta(d_1 \cdots d_p) \times \Delta(e_1 \cdots e_q) \times \{n\} \times \{j\} \in \bigvee_{n=0}^{\infty} \widehat{T}^n \pi^{-1}(\mathcal{B})$$

and thus we see that

$$\mathcal{R} = \bigvee_{n=0}^{\infty} \widehat{T}^n \pi^{-1}(\mathcal{B}). \qquad \square$$

This gives the following theorem.

Theorem 3.5.1. *The dynamical system $(R, \mathcal{R}, \nu, \widehat{T})$ is a version of the natural extension of the dynamical system $([0,1), \mathcal{B}, \mu, T)$. Here, the measure μ, given by $\mu(E) = \nu(\pi^{-1}(E))$ for all measurable sets E, is the ACIM of T and the density of μ is equal to the density from (2.11).*

Proof. The fact that $(R, \mathcal{R}, \nu, \widehat{T})$ is a version of the natural extension of the system $([0,1), \mathcal{B}, \mu, T)$ follows from Remark 3.4.3, the properties of the map π and Lemma 3.5.2. Now for each measurable set $E \in \mathcal{B}([0,1))$, we have

$$\begin{aligned}
\mu(E) &= \nu(\pi^{-1}(E)) \\
&= \frac{1}{\lambda_R(R)}\Big[\lambda(E) + \sum_{n=1}^{\infty} \sum_{\Delta(b_1\cdots b_n)\in D_n} \frac{1}{\beta^n}\lambda(E \cap T^n\Delta(b_1\cdots b_n))\Big] \\
&= \frac{1}{\lambda_R(R)}\int_E \Big[1 + \sum_{n=1}^{\infty} \sum_{\Delta(b_1\cdots b_n)\in D_n} \frac{1}{\beta^n} 1_{T^n\Delta(b_1\cdots b_n)}\Big] d\lambda.
\end{aligned}$$

Here $T^n\Delta(b_1 \cdots b_n)$ has the form $[0, T^m(u-1))$ or $[0, T^m(\beta - u))$ for some $0 \le i < n$. So, the density of the measure μ equals the density from equation (2.11). □

Remark 3.5.2. (i) The above definitions of the space R and the transformation $\widehat{T}$ can be adapted quite easily for the cases illustrated by Figure 3.2(b) and Figure 3.2(c). We let $\Delta(1)$ take the role of $\Delta(0)$ and consider the orbits of the points 1 and $\beta(u-1) - u$. In general, the sets D_n will contain more elements, but since $2 < \beta < 3$, it is immediate that $\lambda_R(R) < \infty$. This shows that we can construct a version of the natural extension of T, also for these two cases.
(ii) On the set of sequences, the transformation $\widehat{T}$ does not behave like the left-shift, but more like a left-shift with a delay. The transformation only shifts a sequence when the corresponding point returns to R_0 and then it shifts a whole block of digits at the same time. The blocks of digits that it shifts are the full subblocks C_i that we constructed earlier.
(iii) Let R_0' be the set obtained from R_0 by removing the set of measure zero of elements which do not return to R_0, i.e., we remove those (x, y) for which $r_1(x, y) = \infty$. Let $\mathcal{W} : R_0' \to R_0'$ be the transformation induced by $\widehat{T}$, i.e., for all $(x, y) \in R_0'$, let

$$\mathcal{W}(x, y) = \widehat{T}^{r_1(x,y)}(x, y, 0, 0).$$

Then the dynamical system $(R_0', \mathcal{B}(R_0'), \lambda \times \lambda, \mathcal{W})$, where $\mathcal{B}(R_0')$ is the Borel σ-algebra on R_0', is isomorphic to the natural extension of a GLS-transformation as defined in Example 3.1.1. By Proposition 1.2.1, this implies that the system $(R_0', \mathcal{B}(R_0'), \lambda \times \lambda, \mathcal{W})$ is Bernoulli.

Using the invariant measure μ, we will give a short proof of the fact that the greedy β-transformations with three digits are exact. Since the full fundamental intervals generate the Borel σ-algebra on the support of the ACIM, the collection of full fundamental intervals satisfies all conditions of Theorem 1.2.3 from Rohlin. Hence, to show that a greedy β-transformation with three digits T is exact, it is enough to show that there exists a universal constant $\gamma > 0$, such that for any full fundamental interval $\Delta(b_1 \cdots b_n)$ and any measurable subset $E \subseteq \Delta(b_1 \cdots b_n)$, we have

$$\mu(T^n E) \leq \gamma \cdot \frac{\mu(E)}{\mu(\Delta(b_1 \cdots b_n))}.$$

To this end, define two constants, $c_1, c_2 > 0$, by

$$c_1 = \frac{1}{\lambda_R(R)} \quad \text{and} \quad c_2 = 1 + \sum_{n=1}^{\infty} \sum_{\Delta(b_1 \cdots b_n) \in B_n} \frac{1}{\beta^n}.$$

Then for all measurable sets E, we have $c_1 \lambda(E) \leq \mu(E) \leq c_1 c_2 \lambda(E)$. Now, let $\Delta(b_1 \cdots b_n)$ be a full fundamental interval of rank n. Then, by (3.2) we have for each measurable set $E \subseteq \Delta(b_1 \cdots b_n)$ that

$$\lambda(T^n E) = \beta^n \lambda(E) = \frac{1}{\lambda(\Delta(b_1 \cdots b_n))} \lambda(E).$$

Now,

$$\begin{aligned} \mu(T^n E) &\leq c_1 c_2 \lambda(T^n E) = c_1 c_2 \frac{\lambda(E)}{\lambda(\Delta(b_1 \cdots b_n))} \\ &\leq c_1 c_2 \frac{\mu(E) c_1 c_2}{c_1 \mu(\Delta(b_1 \cdots b_n))} = c_1 c_2^2 \frac{\mu(E)}{\mu(\Delta(b_1 \cdots b_n))}. \end{aligned}$$

If we take $\gamma = c_1 c_2^2$, then $\gamma > 0$ and

$$\mu(T^n E) \leq \gamma \cdot \frac{\mu(E)}{\mu(\Delta(b_1 \cdots b_n))}.$$

Thus, T is exact and hence mixing of all orders. Furthermore, by definition the natural extension transformation $\widehat{T}$ is a K-automorphism. By Theorem 3.1.2 from Rychlik, it follows immediately that T is weakly Bernoulli.

3.5.2 *An example*

It is time for an example. Let G be the golden mean and take $\beta = G$. Consider the allowable digit set $A = \{0, 1, \frac{4}{3}\}$. The partition $\Delta = \{\Delta(0), \Delta(1), \Delta(\frac{4}{3})\}$ is given by

$$\Delta(0) = \left[0, \frac{1}{\beta}\right), \quad \Delta(1) = \left[\frac{1}{\beta}, \frac{4}{3\beta}\right), \quad \Delta(\frac{4}{3}) = \left[\frac{4}{3\beta}, 1\right)$$

and the transformation then is $Tx = \beta x - i$, if $x \in \Delta(i)$.

The points $x = \frac{1}{3}$ and $x = \beta - \frac{4}{3}$ are of special interest and their orbits under T are as follows.

$$\frac{1}{3}, \quad T\left(\frac{1}{3}\right) = \frac{\beta}{3}, \quad T^2\left(\frac{1}{3}\right) = \frac{\beta^2}{3}, \quad T^3\left(\frac{1}{3}\right) = \frac{1}{3\beta^3},$$

$$T^4\left(\frac{1}{3}\right) = \frac{1}{3\beta^2}, \quad T^5\left(\frac{1}{3}\right) = \frac{1}{3\beta}, \quad T^6\left(\frac{1}{3}\right) = \frac{1}{3},$$

$$\beta - \frac{4}{3}, \quad T\left(\beta - \frac{4}{3}\right) = 1 - \frac{\beta}{3}, \quad T^2\left(\beta - \frac{4}{3}\right) = \frac{2}{3}\beta - \frac{1}{3} \quad T^3\left(\beta - \frac{4}{3}\right) = \frac{1}{3\beta}.$$

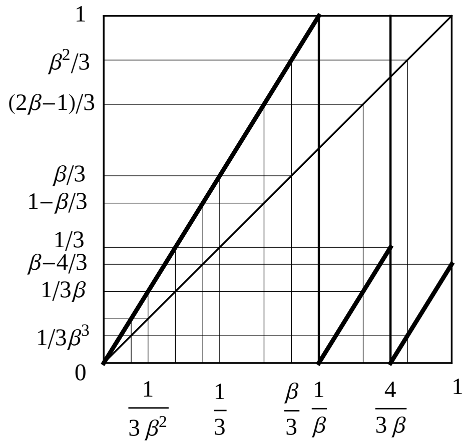


Figure 3.9 The orbits of $\frac{1}{3}$ and $\beta-\frac{4}{3}$ under the greedy β-transformation with $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{0, 1, \frac{4}{3}\}$.

Thus their greedy β-expansions are given by

$$\begin{aligned} \frac{1}{3} &= \sum_{k=1}^{\infty} \frac{d_k^{(1)}}{\beta^k} = .\left(00\frac{4}{3}00\right)^{\omega}, \\ \beta - \frac{4}{3} &= \sum_{k=1}^{\infty} \frac{d_k^{(4/3)}}{\beta^k} = .001\left(0000\frac{4}{3}\right)^{\omega}. \end{aligned}$$

Notice that $\beta - \frac{4}{3}$ would be the image of 1 under T, if T were not restricted to the interval $[0, 1)$. Figure 3.9 shows the orbits of both points under T. For each non-full fundamental interval, $\Delta(b_1 \cdots b_n)$, the set $T^n\Delta(b_1 \cdots b_n)$ is one of the following:

$$\left[0, \beta - \frac{4}{3}\right), \quad \left[0, 1 - \frac{\beta}{3}\right), \quad \left[0, \frac{2\beta-1}{3}\right), \quad \left[0, \frac{1}{3}\right), \quad \left[0, \frac{\beta}{3}\right),$$

$$\left[0, \frac{\beta^2}{3}\right), \quad \left[0, \frac{1}{3\beta^3}\right), \quad \left[0, \frac{1}{3\beta^2}\right), \quad \left[0, \frac{1}{\beta}\right).$$

This means that we could give all the elements of R_n explicitly. We will not do this, since Figure 3.10 speaks for itself. The space R contains all the rectangles shown in Figure 3.10.

To determine the invariant measure of T, we need to project the Lebesgue measure on the rectangles of R onto the first coordinate. Therefore, we need to add up the heights of all the rectangles which have the same interval in the first coordinate. To do this, we only need to determine the total height of the rectangles of the form $[0, \frac{1}{3}) \times [0, \frac{1}{\beta^n})$, since the total height of all the other rectangles in R can be deduced from this. These heights can be found by using the Fibonacci numbers $F(n)$, as defined before. We have $\kappa(1) = 2$

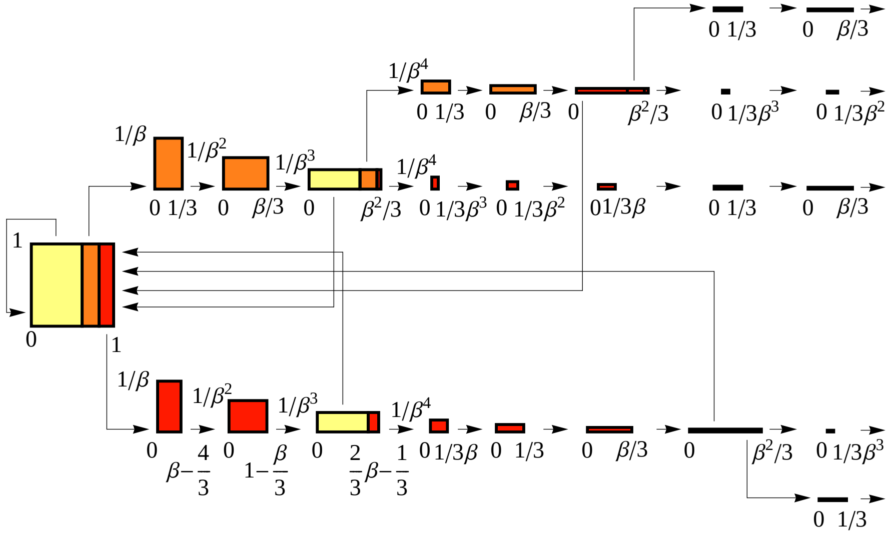


Figure 3.10 The space R consists of all these rectangles. The arrows indicate where the rectangles are mapped under $\widehat{T}$.

and observe that for $n \geq 2$, $\kappa(n) = F(\lfloor \frac{n-1}{3} \rfloor + 2) + F(\lfloor \frac{n-2}{3} \rfloor + 1)$. For $n = 3k+1$, $k \geq 0$, the number of rectangles in R of the form $[0, \frac{1}{3}) \times [0, \frac{1}{\beta^n})$ is equal to $F(k+1)$ and for $n = 3k+2$, $k \geq 1$, this number is equal to $F(k)$. Formula (3.10) gives that the total height of all these rectangle is equal to

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{F(k+1)}{\beta^{3k+1}} + \sum_{k=1}^{\infty} \frac{F(k)}{\beta^{3k+2}} &= \frac{1}{\beta}\left[1 + \sum_{k=1}^{\infty} \frac{\beta F(k+1) + F(k)}{\beta^{3k+1}}\right] \\
&= \frac{1}{\beta}\left[1 + \frac{1}{\sqrt{5}}\left(\beta \sum_{k=1}^{\infty} \frac{1}{\beta^{2k}} + \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{1}{\beta^{2k}}\right)\right] \\
&= \frac{1}{\beta}\left[1 + \frac{1}{\sqrt{5}}\left((\beta + \frac{1}{\beta})(\frac{\beta^2}{\beta^2 - 1} - 1)\right)\right] \\
&= \frac{1}{\beta}\left[1 + \frac{1}{\sqrt{5}}(3 - \beta)\right] = \frac{1}{3}.
\end{aligned}$$

The total height of the rectangles of the form $[0, \frac{\beta}{3}) \times [0, \frac{1}{\beta^n})$ is now equal to $\frac{1}{\beta}$, that of the rectangles of the form $[0, \frac{\beta^2}{3}) \times [0, \frac{1}{\beta^n})$ is $\frac{1}{\beta^2}$, etc. The total height of the rectangles $[0, \frac{1}{3\beta}) \times [0, \frac{1}{\beta^n})$ is given by $\frac{1}{\beta^4} + \frac{1}{\beta^5} = \frac{1}{\beta^3}$. Then the density function of the ACIM of T, $h:[0,1)\to[0,1)$, is given by

$$\begin{aligned} h(x) \quad = \quad & \frac{3}{58-31\beta}\Bigg[\frac{1}{\beta}\,1_{[0,\beta-\frac{4}{3})}(x)+\frac{1}{\beta^2}\,1_{[0,1-\frac{\beta}{3})}(x)+\frac{1}{\beta^3}\,1_{[0,\frac{2}{3}\beta-\frac{1}{3})}(x) \\ & +1_{[0,\frac{1}{3})}(x)+\frac{1}{\beta}\,1_{[0,\frac{\beta}{3})}(x)+\frac{1}{\beta^2}\,1_{[0,\frac{\beta^2}{3})}(x)+\frac{1}{\beta^3}\,1_{[0,\frac{1}{3\beta^3})}(x) \\ & +\frac{1}{\beta^4}\,1_{[0,\frac{1}{3\beta^2})}(x)+\frac{1}{\beta^3}1_{[0,\frac{1}{3\beta})}(x)+1_{[0,1)}(x)\Bigg]. \end{aligned}$$

Final remarks: We have constructed a version of the natural extension of the greedy β-transformation with three arbitrary digits in case the real number $\beta>1$ and the digit set A satisfy condition (3.1). From the natural extension we have obtained an explicit expression for the density function of this transformation. This density function is equal to the density found by Wilkinson in [Wil75]. The transformation T is exact and weakly Bernoulli. A very similar construction can be used to give a version of the natural extension for the transformation in case $A=\{0,1,u\}$ does not satisfy condition (3.1).

Of course, it still remains to find an explicit expression of the form (2.11) for the ACIM of the greedy β-transformation with arbitrary digits, of which the digit set contains more than three digits. If the number of digits, $m+1$, satisfies $m<\beta\le m+1$, the density is given by the density from (2.11). For this case, one could try to generalize the definition of the version of the natural extension that we gave in this section. For the space R we would then have $\kappa(n)\le m^n$ and the finiteness of the measure of R is guaranteed, since $\beta>m$. Recall, however, that Kopf ([Kop90]) and Góra ([Gór09]) gave expressions of a different form for the ACIMs of more general classes of piecewise linear transformations.

The first and second version of Section 3.3 come from [DK]. Everything from Section 3.5 can be found in [DK09a].

CHAPTER 4

ON LAZINESS AND MAKING CHOICES

4.1 We want more

The greedy β-expansion is one way to write numbers in base β and with digits in a certain set, but it is certainly not the only way. In general there are many different expansions for the same number. If $\beta > 1$ is a non-integer and the digit set is $\{0, 1, \ldots, \lfloor\beta\rfloor\}$, then almost all $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ have a continuum of expansions in base β and digits in $\{0, 1, \ldots, \lfloor\beta\rfloor\}$, see [EJK90], [Sid03], [DdV07]. In the greedy expansions each digit is the largest digit possible for that place. In that sense they represent an extreme way of expanding numbers. The other extreme would be to take at each step the smallest digit possible. For the digit set $\{0, 1, \ldots, \lfloor\beta\rfloor\}$, an algorithm for this was first introduced in [EJK90] by Erdös, Joó and Komornik. It is called the *lazy algorithm* and is given recursively as follows. If the first $n-1$ lazy digits of the expansion of $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ are given by $c_1, \ldots, c_{n-1}$, then the n-th digit, c_n, is the smallest element of $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ such that

$$\sum_{k=1}^{n} \frac{c_k}{\beta^k} + \sum_{k=n+1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^k} \geq x.$$

Using this algorithm we can write each $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ as $x = \sum_{k=1}^{\infty} \frac{c_k}{\beta^k}$. This expansion is called the *classical lazy β-expansion* of x. In [DK02b], Dajani and Kraaikamp gave a transformation, L_β, that dynamically generates these lazy expansions. It is defined from the interval $\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ to itself by

$$L_\beta(x) = \beta x - d \quad \text{for } \; x \in \Delta(d),$$

where

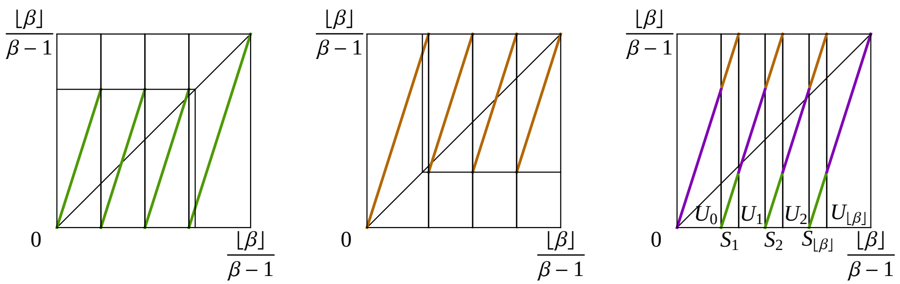
$$\Delta(0) = \left[0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right]$$

and

$$\begin{aligned}\Delta(d) &= \left(\frac{\lfloor\beta\rfloor}{\beta-1}-\frac{\lfloor\beta\rfloor-d+1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}-\frac{\lfloor\beta\rfloor-d}{\beta}\right] \\ &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)}+\frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}+\frac{d}{\beta}\right], \quad 1\le d\le\lfloor\beta\rfloor.\end{aligned}$$

This transformation is the *classical lazy β-transformation*. In [DK02b] it is proven that there is a measure theoretical isomorphism between the greedy and the lazy β-transformations. This isomorphism is given by $\theta : \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \to \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] : x \mapsto \frac{\lfloor\beta\rfloor}{\beta-1} - x$. Thus $\theta \circ T_\beta = L_\beta \circ \theta$. If μ_β denotes the ACIM for T_β, then $\nu_\beta = \mu_\beta \circ \theta^{-1}$ is the ACIM for L_β. As a consequence, the dynamical system of the lazy transformation, $(\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right], \mathcal{B}(\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]), \nu_\beta, L_\beta)$ is ergodic and weakly Bernoulli.

The greedy and lazy expansions of an x represent the two extremes in the sense that one chooses the largest digit in each step and the other one chooses the smallest digit. Looking at the first digit in these expansions, we see that the classical greedy and lazy digit either are the same or they differ by 1, i.e., if the two digits are not the same, then the greedy digit is the lazy digit plus 1. This can also be seen from Figure 4.1, where the two transformations are superimposed.



(a) Greedy (b) Lazy (c) Both

Figure 4.1 The classical greedy and lazy β-transformation for $\beta = \pi$.

We see that on the interval $\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ we can distinguish two types of intervals. There are uniqueness regions, on which the greedy and lazy transformation are equal, and there are switch regions, on which the greedy and lazy transformation assign a different digit. We can give the regions explicitly. For all $\beta > 1$, there are $\lfloor\beta\rfloor + 1$ uniqueness regions, $U_0, \ldots, U_{\lfloor\beta\rfloor}$ and $\lfloor\beta\rfloor$ switch regions, $S_1, \ldots, S_{\lfloor\beta\rfloor}$, where

$$\begin{aligned} U_0 &= \left[0, \frac{1}{\beta}\right), \quad U_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor - 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right], \\ U_i &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}, \frac{i+1}{\beta}\right), \quad 1 \le i \le \lfloor\beta\rfloor - 1, \\ S_i &= \left[\frac{i}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{i-1}{\beta}\right], \quad 1 \le i \le \lfloor\beta\rfloor. \end{aligned}$$

On S_i the greedy map assigns the digit i, while the lazy map assigns the digit $i-1$. On U_i both maps assign the same digit i. In [DdV07] Dajani and De Vries defined a transformation that generates in a random way all possible expansions in base β and with digits in $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ of an $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$. This transformation is defined on the product space $\{0,1\}^\omega \times \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$. The idea is as follows. If an x lies in a uniqueness region, there is only one digit that can be assigned. If an x lies in a switch region, then we have a choice between two digits. We let the first element of the sequence $w \in \{0,1\}^\omega$ decide. It is like a coin toss. If the first element of w is a 1, we use the greedy digit and apply the greedy map, and if not, we use the lazy digit and apply the lazy map. Then we shift the sequence w one place to the left and continue. We can make this precise in the following way. Let $\sigma : \{0,1\}^\omega \to \{0,1\}^\omega$ be the left-shift. The transformation K_β from $\{0,1\}^\omega \times \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ to itself is given by

$$K_\beta(w, x) = \begin{cases} (w, \beta x - i), & \text{if } x \in U_i,\ 0 \le i \le \lfloor\beta\rfloor, \\ (\sigma(w), \beta x - i), & \text{if } x \in S_i \text{ and } w_1 = 1,\ 1 \le i \le \lfloor\beta\rfloor, \\ (\sigma(w), \beta x - i + 1), & \text{if } x \in S_i \text{ and } w_1 = 0,\ 1 \le i \le \lfloor\beta\rfloor. \end{cases}$$

The digits are defined by

$$d_1 = d_1(w, x) = \begin{cases} i, & \text{if } x \in U_i,\ 0 \le i \le \lfloor\beta\rfloor, \\ & \text{ or } (w, x) \in \{w_1 = 1\} \times S_i,\ 1 \le i \le \lfloor\beta\rfloor, \\ i - 1, & \text{if } (w, x) \in \{w_1 = 0\} \times S_i,\ 1 \le i \le \lfloor\beta\rfloor, \end{cases}$$

and for $n \ge 1$, $d_n = d_n(w, x) = d_1(K_\beta^{n-1}(w, x))$. In [DdV07] it is shown that this transformation generates all possible expansions of an $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ in base β and with digits in $\{0, 1, \ldots, \lfloor\beta\rfloor\}$. Note that an element x has a unique expansion if and only if the orbit of x under the map T_β hits only the uniqueness regions. Furthermore, in [DdV07] it is shown that there exists a unique K_β-invariant measure of maximal entropy ρ_β such that the system $\left(\{0,1\}^\omega \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}], \mathcal{F} \times \mathcal{B}(\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]), \rho_\beta, K_\beta\right)$ is isomorphic to the uniform Bernoulli shift on $\lfloor\beta\rfloor+1$ symbols, where $\mathcal{F}$ is the product σ-algebra on $\{0,1\}^\omega$.

In this chapter we will define the lazy algorithm and the lazy transformation for arbitrary allowable digit sets. We will show that in general, there is a whole family of transformations that generate β-expansions with digits in an allowable set A. This is done in Section 4.2. In Section 4.3, we define a random transformation, that generates all possible expansions of a given x in a certain base and allowable digit set. We show that also in this case the dynamical system of the random transformation is isomorphic to a Bernoulli shift. The last section of this chapter discusses some known results on what happens when the digit set is not allowable. This section does not contain new results and is added only for completeness.

4.2 Not everything is greedy or lazy

For general digit sets, we can also define, recursively as well as dynamically, a lazy algorithm, generating β-expansions that have in each place the smallest digit possible. Given a real number $\beta > 1$ and an allowable digit set $A = \{0, a_1, \dots, a_m\}$, we first define the lazy expansions recursively as follows. Let $x \in \big[0, \frac{a_m}{\beta-1}\big]$, and suppose the digits $c_1 = c_1(x), \dots, c_{n-1} = c_{n-1}(x)$ are already defined. Then $c_n = c_n(x)$ is the smallest element of A such that

$$x \leq \sum_{k=1}^{n-1} \frac{c_k}{\beta^k} + \frac{c_n}{\beta^n} + \sum_{k=n+1}^{\infty} \frac{a_m}{\beta^k}. \tag{4.1}$$

It is not yet obvious that this leads to an expansion, but we will use this notation formally in order to derive a dynamical definition where it will be clear that such an algorithm leads to an expansion. Suppose that we have a lazy expansion for x, i.e., $x = \sum_{k=1}^{\infty} \frac{c_k}{\beta^k}$ where the digits c_k are the lazy digits. Rewriting condition (4.1), one sees that for $1 \leq i \leq m$,

$$\begin{aligned} c_n = a_i \quad &\Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{c_k}{\beta^k} + \frac{a_{i-1}}{\beta^n} + \sum_{k=n+1}^{\infty} \frac{a_m}{\beta^k} < x \leq \sum_{k=1}^{n-1} \frac{c_k}{\beta^k} + \frac{a_i}{\beta^n} + \sum_{k=n+1}^{\infty} \frac{a_m}{\beta^k} \\ &\Leftrightarrow \quad \frac{a_m}{\beta-1} - \frac{a_m - a_{i-1}}{\beta} < \sum_{k=1}^{\infty} \frac{c_{n-1+k}}{\beta^k} \leq \frac{a_m}{\beta-1} - \frac{a_m - a_i}{\beta}, \end{aligned}$$

and $c_n = 0$ if and only if

$$0 \leq \sum_{k=1}^{\infty} \frac{c_{n-1+k}}{\beta^k} \leq \frac{a_m}{\beta-1} - \frac{a_m}{\beta}.$$

As in the greedy case, we want to define a transformation L on $\big[0, \frac{a_m}{\beta-1}\big]$ such that if $x = \sum_{k=1}^{\infty} \frac{c_k}{\beta^k}$ is the lazy expansion of x, then $L^{n-1}x = \sum_{k=1}^{\infty} \frac{c_{n-1+k}}{\beta^k}$. In view of the above, we define the lazy transformation as follows.

Definition 4.2.1. Let $\beta > 1$ and let $A = \{a_0 = 0, a_1, \dots, a_m\}$ be an allowable digit set for β. The *lazy transformation* $L = L_{\beta,A}$ *with digit set* A is given by

$$Lx = \begin{cases} \beta x, & \text{if } x \in \left[0, \dfrac{a_m}{\beta-1} - \dfrac{a_m}{\beta}\right], \\[2ex] \beta x - a_i, & \text{if } x \in \left(\dfrac{a_m}{\beta-1} - \dfrac{a_m - a_{i-1}}{\beta}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_i}{\beta}\right], \\ & \text{for } 1 \le i \le m. \end{cases}$$

The transformation in Figure 4.2 is the lazy transformation that has the same β and the same digit set A as the greedy transformation in Figure 2.2. Note that $L\left(\left[0, \frac{a_m}{\beta-1} - \frac{a_m}{\beta}\right]\right) = \left[0, \frac{a_m}{\beta-1}\right]$. Furthermore, for $1 \le i \le m$ one has

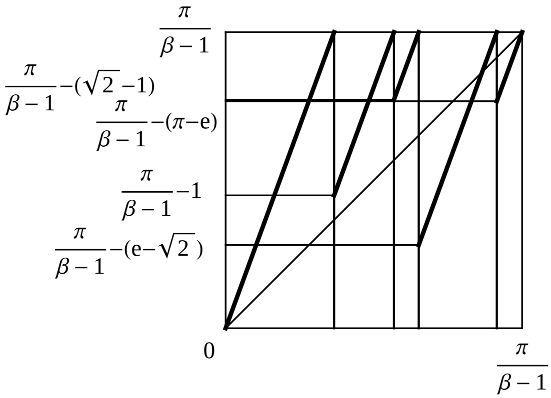


Figure 4.2 The lazy β-transformation for $\beta = 1 + \sqrt{3}$ and $A = \{0, 1, \sqrt{2}, e, \pi\}$.

$$L\left(\left(\frac{a_m}{\beta-1} - \frac{a_m - a_{i-1}}{\beta}, \frac{a_m}{\beta-1} - \frac{a_m - a_i}{\beta}\right]\right) = \left(\frac{a_m}{\beta-1} - (a_i - a_{i-1}), \frac{a_m}{\beta-1}\right]$$

and since A is allowable, then for each $1 \le i \le m$,

$$\left(\frac{a_m}{\beta-1} - (a_i - a_{i-1}), \frac{a_m}{\beta-1}\right] \subseteq \left[0, \frac{a_m}{\beta-1}\right].$$

This shows that L maps the interval $\left[0, \frac{a_m}{\beta-1}\right]$ onto itself. Let

$$c_1 = c_1(x) = \begin{cases} 0, & \text{if } x \in \left[0, \dfrac{a_m}{\beta-1} - \dfrac{a_m}{\beta}\right], \\[2ex] a_i, & \text{if } x \in \left(\dfrac{a_m}{\beta-1} - \dfrac{a_m - a_{i-1}}{\beta}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_i}{\beta}\right], \\ & \text{for } 1 \le i \le m, \end{cases}$$

and for $n \ge 1$, set $c_n = c_n(x) = c_1(L^{n-1}x)$. Then, $Lx = \beta x - c_1$, and for any $n \ge 1$,

$$x = \sum_{k=1}^{n} \frac{c_k}{\beta^k} + \frac{L^n x}{\beta^n}.$$

Letting $n \to \infty$, it is easily seen that $x = \sum_{k=1}^{\infty} \frac{c_k}{\beta^k}$, with c_k satisfying (4.1). From the definition of the map L, we see that the point 0 is the only point in the interval $\left[0, \frac{a_m}{\beta-1}\right]$ whose lazy expansion eventually ends in the sequence $00\cdots$.

There is a relation between the greedy and lazy β-transformations with arbitrary digit sets. This is given in Proposition 4.2.1. For a digit set $A = \{a_0 = 0, a_1, \ldots, a_m\}$, define the digit set $\bar{A} = \{\bar{a}_0 = 0, \bar{a}_1, \ldots, \bar{a}_m\}$, by $\bar{a}_i = a_m - a_{m-i}$. Notice that $\bar{a}_m = a_m$ and $\bar{a}_{i+1} - \bar{a}_i = a_{m-i} - a_{m-(i+1)}$, for $0 \le i \le m-1$. Thus, A is allowable if and only if $\bar{A}$ is allowable. We have the following proposition.

Proposition 4.2.1. *Let $A = \{0, a_1, \ldots, a_m\}$ be an allowable digit set, and let $T = T_{\beta,A}$ and $L = L_{\beta,\bar{A}}$ be the greedy and lazy β-transformations with digit sets A and $\bar{A}$, respectively. Define $\theta : \left[0, \frac{a_m}{\beta-1}\right] \to \left[0, \frac{a_m}{\beta-1}\right]$ by*

$$\theta(x) = \frac{a_m}{\beta - 1} - x.$$

Then, θ is a continuous bijection satisfying $L \circ \theta = \theta \circ T$.

Proof. It is clear that θ is a continuous bijection. It remains to show that $L \circ \theta = \theta \circ T$. To this end, let $x \in \left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\right)$, then $Tx = \beta x - a_i$ and

$$\theta(Tx) = \frac{a_m}{\beta - 1} - \beta x + a_i.$$

On the other hand,

$$\theta(x) \in \left(\frac{\bar{a}_m}{\beta - 1} - \frac{\bar{a}_m - \bar{a}_{m-(i+1)}}{\beta}, \frac{\bar{a}_m}{\beta - 1} - \frac{\bar{a}_m - \bar{a}_{m-i}}{\beta}\right].$$

Thus,

$$L\theta(x) = \beta\theta(x) - \bar{a}_{m-i} = \frac{a_m}{\beta - 1} - \beta x + a_i = \theta(Tx).$$

A similar proof works in case $x \in \left[\frac{a_m}{\beta}, \frac{a_m}{\beta-1}\right]$. □

Remark 4.2.1. From Proposition 4.2.1, we see that if $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$ is the greedy expansion of x with digits in A, then $\theta(x) = \sum_{k=1}^{\infty} \frac{\bar{b}_k}{\beta^k}$ is the lazy expansion of $\theta(x)$ with digits in $\bar{A}$.

For $1 \le i \le m$, let $\bar{y}_i$ be the values of L of the limits from the right to the points of discontinuity of L, i.e., $\bar{y}_i = \frac{\bar{a}_m}{\beta-1} - (\bar{a}_i - \bar{a}_{i-1})$. As a direct consequence of Proposition 4.2.1 and Theorem 2.3.1, we get the following corollary about the ACIM of L. Recall that i_0 was defined in (2.9) for the greedy transformation by

$$i_0 = \min\{i \mid T[0, y_i] \subseteq [0, y_i]\ \lambda \text{ a.e.}\}.$$

Corollary 4.2.1. *Let L be the lazy transformation for $\beta > 1$ and allowable digit set $\bar{A} = \{\bar{a}_0 = 0, \bar{a}_1, \ldots, \bar{a}_m\}$. Let T be the greedy transformation for the same $\beta > 1$ and with allowable digit set $A = \{a_0 = 0, a_1, \ldots, a_m\}$, such that $a_i = \bar{a}_m - \bar{a}_{m-i}$. Then there exists a unique absolutely continuous invariant measure ν for L, that is ergodic. Let i_0 be defined for the greedy transformation as in equation (2.9). Then the support of the measure ν is given by the interval $\left[\bar{y}_{m-i_0+1}, \frac{\bar{a}_m}{\beta-1}\right]$.*

Proof. By Theorem 2.3.1 we have that the interval $[0, y_{i_0}]$ is the support of the invariant measure for the greedy transformation. Let μ be the ACIM for T. Since T and L are isomorphic with isomorphism θ as given in Proposition 4.2.1, L also has a unique absolutely continuous invariant measure, given by $\nu = \mu \circ \theta^{-1}$. This is an ergodic measure and its support is given by

$$\theta([0, y_{i_0}]) = \left[\bar{y}_{m-i_0+1}, \frac{\bar{a}_m}{\beta - 1}\right]. \qquad \square$$

The greedy and lazy transformations are two extreme ways to generate expansions, but they are certainly not the only transformations with this property. To see this, we superimpose the greedy and lazy transformation for the same digit set, as was done in Section 4.1. In Figure 4.3, we see the greedy and lazy transformations for $\beta = 1 + \sqrt{2}$ and $A = \{0, \frac{\pi}{\sqrt{2}}, e, \pi\}$ and their superposition. We see that there are regions in which the greedy and lazy transformation coincide, but there are also regions in which we can choose between a number of different digits. We will call the regions where the two transformations coincide *uniqueness regions*, since the digits assigned there are completely determined. The regions in which we can make a choice are called *switch regions*. In Figure 4.3, there are three uniqueness regions and three switch regions. On two switch regions we have a choice of two possible digits, and on one switch region we have a choice of three possible digits. The boundaries of these regions are given by the greedy and lazy transformations. Note that for the digit set $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ one has only two choices in each switch region, the greedy and the lazy digit. For a general allowable set it is sometimes possible to choose a digit which is neither greedy nor lazy. To describe the general situation, we first define the points that are given by the greedy and lazy transformation.

Definition 4.2.2. For $1 \leq i \leq m$, let the *greedy partition points* be given by

$$g_i = \frac{a_i}{\beta}$$

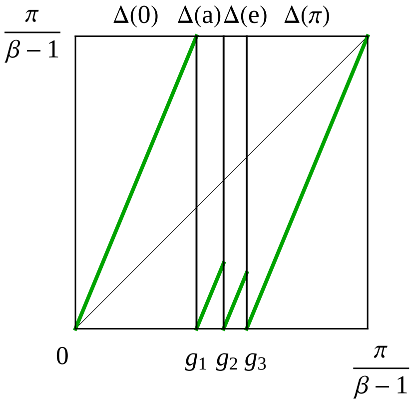
and the *lazy partition points* by

$$l_i = \frac{a_m}{\beta - 1} - \frac{a_m - a_{i-1}}{\beta}.$$

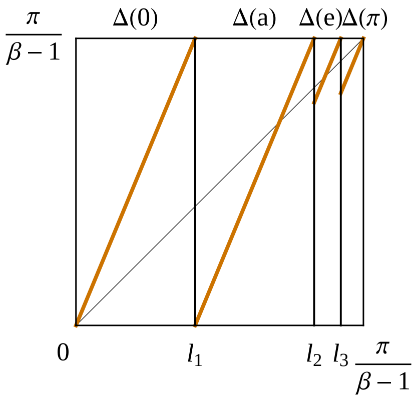
We set $l_0 = g_0 = 0$ and $l_{m+1} = g_{m+1} = \frac{a_m}{\beta - 1}$.

Notice that $g_i \leq l_i$ for all $0 \leq i \leq m+1$. Furthermore, on $[g_i, g_{i+1})$ the greedy transformation T is given by $Tx = \beta x - a_i$, and on $(l_i, l_{i+1}]$ the lazy transformation L is given by $Lx = \beta x - a_i$. If $l_i = g_{i+j}$ for some i, j with $i > 0$, $i + j < m + 1$, then any expansion ending in the digits $a_{i-1} a_m a_m \cdots$ has a corresponding representation ending in the digits $a_{i+j} 0 0 \cdots$.

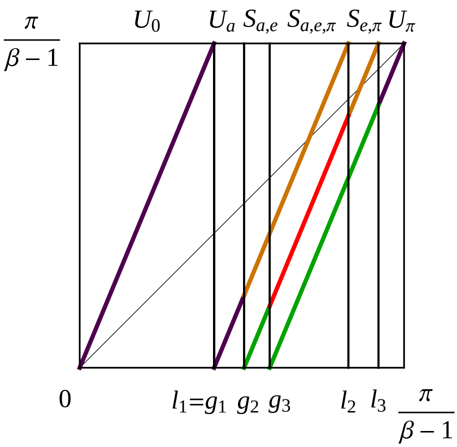
Using the points l_i and g_i, for $0 \leq i \leq m+1$, we can make a partition of the interval $\left[0, \frac{a_m}{\beta - 1}\right]$ in the following way. Consider the set of all greedy and lazy partition points. If for some $i, j \geq 1$ we have $l_i = g_{i+j}$, then we take the point once, and we consider it as a greedy partition point. Suppose that we are left with N points. Let $\{p_n \mid 0 \leq n \leq N - 1\}$ be the ordered sequence obtained



(a) Greedy



(b) Lazy



(c) Both

Figure 4.3 The greedy and lazy β-transformations for $\beta = 1 + \sqrt{2}$ and $A = \{0, \pi/\sqrt{2}, e, \pi\}$. In the picture, a stands for $\pi/\sqrt{2}$.

when the greedy and the lazy partition points are written in increasing order. Note that

$$p_0 = l_0 = g_0 = 0 \quad \text{and} \quad p_{N-1} = l_{m+1} = g_{m+1} = \frac{a_m}{\beta - 1}.$$

Now consider the partition α consisting of the intervals of which the endpoints are two consecutive elements of this sequence, in such a way that the left endpoint of such an interval is included if it is given by a greedy partition point and excluded if it is given by a lazy partition point. Similarly, the right endpoint is excluded if it is given by a greedy partition point and included if it is given by a lazy partition point. We will include 0 in the first interval and $\frac{a_m}{\beta-1}$ in the last interval.

Example 4.2.1. Let $\beta = 1 + \sqrt{2}$ and $A = \{0, \pi/\sqrt{2}, e, \pi\}$ as in Figure 4.3. The greedy and lazy partition points are given by

$$\begin{aligned} g_1 &= \tfrac{\pi}{\sqrt{2}\beta} = \tfrac{\pi}{\beta(\beta-1)}, & l_1 &= \tfrac{\pi}{\beta-1} - \tfrac{\pi}{\beta} = \tfrac{\pi}{\beta(\beta-1)}, \\ g_2 &= \tfrac{e}{\beta}, & l_2 &= \tfrac{\pi}{\beta-1} - \tfrac{\pi-\pi/\sqrt{2}}{\beta}, \\ g_3 &= \tfrac{\pi}{\beta}, & l_3 &= \tfrac{\pi}{\beta-1} - \tfrac{\pi-e}{\beta}. \end{aligned}$$

The partition α is then

$$\alpha = \{[0, g_1), [g_1, g_2), [g_2, g_3), [g_3, l_2], (l_2, l_3], \big(l_3, \tfrac{\pi}{\beta-1}\big]\}.$$

We now identify explicitly the intervals that are the uniqueness regions and those that are the switch regions. If there is a $1 \le i \le m-1$ such that $l_i < g_{i+1}$, then since $g_j \le l_j$ for each j, we know that the interval (l_i, g_{i+1}) must belong to α. On such an interval, the greedy and lazy transformation coincide. This interval is therefore a uniqueness region, and we will call it U_{a_i}. Furthermore, we will write $[p_0, p_1) = [0, g_1) = U_{a_1}$ and $(p_{N-2}, p_{N-1}] = (l_m, \frac{a_m}{\beta-1}] = U_{a_m}$, since the two transformations always coincide in the first and last interval. In all the other cases the interval is a switch region. A switch region with endpoints p_n and p_{n+1} is called $S_{a_i,\dots,a_{i+j}}$ if $i+j = \max\{k \mid g_k \le p_n\}$ and $i = \max\{k \mid l_k \le p_n\}$. So, the endpoints of $S_{a_i,\dots,a_{i+j}}$ are given by $\max\{g_{i+j}, l_i\}$ and $\min\{g_{i+j+1}, l_{i+1}\}$. In this way, every element of the partition α is either a uniqueness region or a switch region.

Example 4.2.2. If we take β and A as in the previous example, we have the following uniqueness and switch regions:

$$\begin{aligned} P_1 &= [0, g_1) = U_0, & P_2 &= [g_1, g_2) = U_{\pi/\sqrt{2}}, \\ P_3 &= [g_2, g_3) = S_{\pi/\sqrt{2},e}, & P_4 &= [g_3, l_2] = S_{\pi/\sqrt{2},e,\pi}, \\ P_5 &= (l_2, l_3] = S_{e,\pi}, & P_6 &= (l_3, \tfrac{\pi}{\beta-1}] = U_\pi. \end{aligned}$$

In two special cases we can explicitly give the locations of the uniqueness regions and switch regions.

Proposition 4.2.2. *Let $\beta > 1$ and let $A = \{0, a_1, \dots, a_m\}$ be an allowable digit set.*

(i) If $m = \lfloor \beta \rfloor$, then $g_i \le l_i \le g_{i+1}$ for each $1 \le i \le m-1$, i.e., the greedy and lazy partition points alternate.

(ii) If $\beta < 2$, then $g_m < l_1$, i.e., the last greedy partition point is strictly smaller than the first lazy partition point.

Proof. For (i), since $g_i \le l_i$ for all i, we only have to check that $l_i \le g_{i+1}$ for all $1 \le i \le m-1$. First observe that for $1 \le i \le m-1$ we have

$$a_{i+1} - a_{i-1} = a_m - \sum_{k=0}^{i-2}(a_{k+1} - a_k) - \sum_{k=i+1}^{m-1}(a_{k+1} - a_k).$$

By condition (2.5) we get

$$\begin{aligned} a_{i+1} - a_{i-1} &\geq a_m - (i-1)\frac{a_m}{\beta - 1} - (m-i-1)\frac{a_m}{\beta-1} \\ &= [(\beta-1)-(m-2)]\,\frac{a_m}{\beta-1} \geq \frac{a_m}{\beta-1}, \end{aligned}$$

since $m = \lfloor \beta \rfloor$. So

$$\frac{a_{i+1} - a_{i-1}}{\beta} \geq \frac{a_m}{\beta-1} - \frac{a_m}{\beta},$$

and therefore also

$$l_i = \frac{a_m}{\beta-1} - \frac{a_m - a_{i-1}}{\beta} \leq \frac{a_{i+1}}{\beta} = g_{i+1}.$$

For (ii), just notice that $\beta - 1 < 1$, so that $a_m < \dfrac{a_m}{\beta-1}$. This means that

$$g_m = \frac{a_m}{\beta} < \frac{a_m}{\beta-1} - \frac{a_m}{\beta} = l_1.$$

□

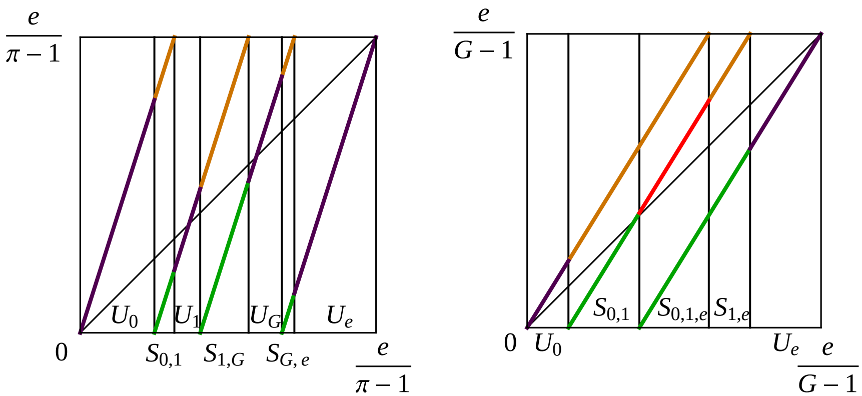
Remark 4.2.2. Notice that if the greedy and lazy partition points are all different, then the first part of the proposition above implies that the uniqueness regions and switch regions alternate in the case $m = \lfloor \beta \rfloor$. The second part of the proposition states that if $\beta < 2$, then we have only two uniqueness regions, namely U_0 and U_{a_m}, and all the other elements of the partition α are switch regions. See Figure 4.4 for examples.

Points in a switch region have more than one expansion. So, if there is a switch region, many points in $\left[0, \frac{a_m}{\beta-1}\right]$ will have many expansions. The only way to have no switch regions is, if for each $0 \leq i \leq m-1$, we have $a_{i+1} - a_i = \frac{a_m}{\beta-1}$, since this implies that $g_i = l_i$ for all i. In that case only the points which eventually get mapped to one of the points g_i, $1 \leq i \leq m$, have more than one expansion and those points have exactly two expansions. This is what happens in the r-adic case, since then $a_m = r-1$ and $a_{i+1} - a_i = 1$ for all i.

If for a certain $\beta > 1$ and allowable digit set $A = \{0, a_1, \ldots, a_m\}$ there is a switch region, then we immediately obtain a whole family of transformations that can generate β-expansions with digits in A for the complete interval $\left[0, \frac{a_m}{\beta-1}\right]$. These transformations are given as follows. If there is a switch region for the digit a_i, then the interval (g_{i+1}, l_{i+1}) is not empty. Choose a $\gamma_{i+1} \in [g_{i+1}, l_{i+1}]$. Do this for all digits, in such a way that $\gamma_i < \gamma_{i+1}$. If there is no switch region for a_i, then $g_{i+1} = l_{i+1}$. Set $\gamma_{i+1} = g_{i+1}$. Let $\gamma_0 = 0$ and $\gamma_{m+1} = \frac{a_m}{\beta-1}$ and set $\gamma = (\gamma_0, \ldots, \gamma_{m+1})$. For $x \in (\gamma_i, \gamma_{i+1})$, $0 \leq i \leq m$, set $d_1(x) = a_i$. Let $d_1(\gamma_0) = 0$, $d_1(\gamma_{m+1}) = a_m$ and for each $1 \leq i \leq m$, make a choice for $d_1(\gamma_i)$ between a_{i-1} and a_i. Set

$$\Delta(a_i) = \left\{ x \in \left[0, \frac{a_m}{\beta-1}\right] \,\middle|\, d_1(x) = a_i \right\},$$

and let the transformation T_γ be defined from the interval $\left[0, \frac{a_m}{\beta-1}\right]$ to itself by $T_\gamma x = \beta x - d_1(x)$. Then $T_\gamma \Delta(a_i) \subseteq \left[0, \frac{a_m}{\beta-1}\right]$ for all i and T_γ is surjective.



(a) $\beta = \pi$, $A = \{0, 1, G, e\}$ (b) $\beta = G$, $A = \{0, 1, e\}$

Figure 4.4 The two cases from Proposition 4.2.2. Here, $G = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Note that the sets $\Delta(a_i)$ are intervals and $\bigcup_{i=0}^{m} \Delta(a_i) = \left[0, \frac{a_m}{\beta-1}\right]$. They are the fundamental intervals for the transformation T_γ. For $n \geq 1$, set $d_n(x) = d_1(T_\gamma^{n-1}x)$. Then,

$$x = \frac{d_1}{\beta} + \frac{T_\gamma x}{\beta} = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \frac{T_\gamma^2 x}{\beta^2} = \cdots = \sum_{k=1}^{n} \frac{d_k}{\beta^k} + \frac{T_\gamma^n x}{\beta^k}.$$

Since $T_\gamma^n x \in \left[0, \frac{a_m}{\beta-1}\right]$ for all $n \geq 1$, we have $x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}$. Thus, T_γ generates β-expansions with digits in A. Figure 4.5 shows such a transformation T_γ.

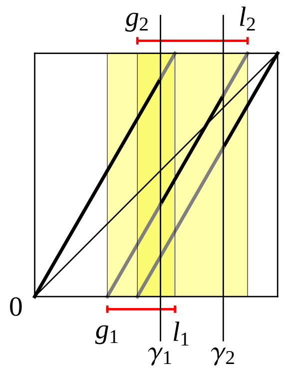


Figure 4.5 A transformation T_γ for $\beta = \sqrt{3}$ and $A = \{0, 1, \sqrt{2}\}$.

Remark 4.2.3. (i) Results from [LY73] by Lasota and Yorke establish the existence of an ACIM for the transformations T_γ. In general, however, such an ACIM does not need to be unique.
(ii) A very similar class of transformations was introduced in [FR08] by Frank

and Robinson. They called them *generalized β-transformations* and used these transformation to prove certain properties of tiling substitutions.

4.3 Roll the dice

In this section, we will define a transformation whose iterates generate all possible β-expansions with allowable digit set A. In order to do so, we will construct a random procedure similar to the one described in Section 4.1. We have the following lemma.

Lemma 4.3.1. *Suppose $x \in \left[0, \frac{a_m}{\beta-1}\right]$ can be written in the form $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$, where $b_k \in A$ for each $k \geq 1$. One has the following.*

(a) If $x \in U_{a_i}$, then $b_1 = a_i$.

(b) If $x \in S_{a_i,\ldots,a_{i+j}}$ and $x \notin \{l_i \,|\, 1 \leq i \leq m\} \cap \{g_i \,|\, 1 \leq i \leq m\}$, then $b_1 \in \{a_i, \ldots, a_{i+j}\}$.

(c) If $x = l_i = g_{i+j}$, then $b_1 \in \{a_{i-1}, \ldots, a_{i+j}\}$.

Proof. Since the proofs of the three statements are very similar, we will only prove the first one. Suppose that $x \in U_{a_i}$ and $b_1 = a_\ell < a_i$. Then $a_i \neq 0$, so the left endpoint of U_{a_i} is given by l_i, and this point itself is not included in the interval. Furthermore,

$$x \leq \frac{a_\ell}{\beta} + \sum_{k=2}^{\infty} \frac{a_m}{\beta^k} = \frac{a_\ell}{\beta} + \frac{a_m}{\beta(\beta-1)} = \frac{a_m}{\beta-1} - \frac{a_m - a_\ell}{\beta} = l_{\ell+1} \leq l_i,$$

contradicting the fact that $x > l_i$. If on the other hand $b_1 = a_\ell > a_i$, then $i \neq m$. The right endpoint of U_{a_i} is then given by g_{i+1}, which itself is not included in the interval. We have $x \geq \frac{a_\ell}{\beta} = g_\ell \geq g_{i+1}$, which also yields a contradiction. Thus, $b_1 = a_i$. □

Remark 4.3.1. Note that in the above lemma, if $x = l_i = g_{i+j}$ and $b_1 = a_{i-1}$, then $b_n = a_m$ for all $n \geq 2$. Hence the given expansion is the lazy expansion. Similarly, if $b_1 = a_{i+j}$, then $b_n = 0$ for all $n \geq 2$. In defining the partition α, we treat these points as greedy partition points. As a consequence, expansions ending in $a_{i-1}a_m a_m \cdots$ cannot be generated dynamically in the random system that we will define in this section. However, we can keep in mind that if for some i, j we have $g_{i+j} = l_i$, then whenever we see an expansion of the form

$$x = \sum_{k=1}^{j} \frac{b_k}{\beta^k} + \frac{a_{i+j}}{\beta^{j+1}},$$

where $b_k \in A$ for each $1 \leq k \leq j$, then the same element x can also be written as

$$x = \sum_{k=1}^{j} \frac{b_k}{\beta^k} + \frac{a_{i-1}}{\beta^{j+1}} + \sum_{k=j+2}^{\infty} \frac{a_m}{\beta^k}.$$

This will ease the exposition.

We will now define a transformation generating random β-expansions in the following way. If an element x lies in a uniqueness region, just assign the digit given by the greedy transformation. To each of the switch regions, we assign a die with an appropriate number of sides. For example, to the switch region $S_{a_i,\dots,a_{i+j}}$ we assign a $(j+1)$-sided die with the numbers $i,\dots,i+j$ on it. If x lies in a switch region, we throw the corresponding die and let the outcome determine the digit we choose.

Suppose α contains Q switch regions. For each $1 \le q \le Q$, if the q-th switch region S_q is given by $S_{a_i,\dots,a_{i+j}}$, let $A_q = \{a_i,\dots,a_{i+j}\}$ and define the set $\Omega^{(q)}$ by $\Omega^{(q)} = \{i,\dots,i+j\}^\omega$. Let each of the sets $\Omega^{(q)}$ be equipped with the product σ-algebra, and let $\sigma^{(q)}$ be the left-shift on $\Omega^{(q)}$. Elements of $\Omega^{(q)}$ indicate a series of outcomes of the die associated with the region S_q and in this manner specify which digit we choose, each time an element hits S_q. Let $\Omega = \prod_{q=1}^{Q}\Omega^{(q)}$ and define the left-shift on the q-th sequence by

$$\sigma_q : \Omega \to \Omega : (w^{(1)},\dots,w^{(Q)}) \mapsto (w^{(1)},\dots,w^{(q-1)},\sigma^{(q)}(w^{(q)}),w^{(q+1)},\dots,w^{(Q)}).$$

On the set $\Omega \times \left[0,\frac{a_m}{\beta-1}\right]$ we define the function $K = K_{\beta,A}$ as follows:

$$K(w,x) = \begin{cases} (w, \beta x - d_1(w,x)), & \text{if } x \in U_{a_i},\ 0 \le i \le m, \\ \\ (\sigma_q(w), \beta x - d_1(w,x)), & \text{if } x \in S_q,\ 1 \le q \le Q, \end{cases}$$

where the sequence of digits $(d_n(w,x))_{n\ge 1}$ is given by

$$d_1(w,x) = \begin{cases} a_j, & \text{if } x \in U_{a_j},\ 0 \le j \le m, \\ \\ a_{w_1^{(q)}}, & \text{if } x \in S_q,\ 1 \le q \le Q, \end{cases}$$

and $d_n(w,x) = d_1(K^{n-1}(w,x))$ for $n \ge 2$. To see how iterations of K generate β-expansions, we let $\pi_2 : \Omega \times \left[0,\frac{a_m}{\beta-1}\right] \to \left[0,\frac{a_m}{\beta-1}\right]$ be the canonical projection onto the second coordinate. Then

$$\pi_2\left(K^n(w,x)\right) = \beta^n x - \beta^{n-1}d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$x = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_2\left(K^n(w,x)\right)}{\beta^n}.$$

Since $\pi_2\left(K^n(w,x)\right) \in \left[0,\frac{a_m}{\beta-1}\right]$, it follows that

$$x - \sum_{k=1}^{n}\frac{d_k}{\beta^k} = \frac{\pi_2\left(K^n(w,x)\right)}{\beta^n} \to 0 \qquad \text{as } n \to \infty.$$

This shows that for all $w \in \Omega$ and for all $x \in \left[0,\frac{a_m}{\beta-1}\right]$ one has that

$$x = \sum_{k=1}^{\infty}\frac{d_k}{\beta^k} = \sum_{k=1}^{\infty}\frac{d_k(w,x)}{\beta^k}.$$

The random procedure just described shows that with each $w \in \Omega$ there corresponds an algorithm that produces expansions in base β. Furthermore, if we identify the point (w, x) with $(w, d_1(w, x)d_2(w, x)\cdots))$, then the action of K on the second coordinate corresponds to the left-shift.

The following theorem states that all β-expansions of an element x can be generated, using a certain w, except when $x = l_i = g_{i+j}$, in which case the lazy expansion of x is not generated by K. As stated in Remark 4.3.1, we disregard this case.

Theorem 4.3.1. *Suppose $x \in \left[0, \frac{a_m}{\beta-1}\right]$ can be written as $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$, with $b_k \in A$ for all $k \geq 1$. Then there exists a $w \in \Omega$ such that $b_n = d_n(w, x)$ for all $n \geq 1$.*

Proof. This proof goes by induction on the number of digits of each $w^{(q)}$ that are determined. First, define the numbers $\{x_n\}_{n\geq 1}$ by $x_n = \sum_{k=1}^{\infty} \frac{b_{k+n-1}}{\beta^k}$, and for $1 \leq q \leq Q$ let the set $\{\ell_n^{(q)}(x)\}_{n\geq 1}$ be given by

$$\ell_n^{(q)}(x) = \sum_{k=1}^{n} 1_{S_q}(x_k).$$

These numbers keep track of the number of times that the orbit of x hits the corresponding switch region.

- If $x \in U_{a_i}$, then by Lemma 4.3.1 we know that $b_1 = a_i$. Then for all $1 \leq k \leq Q$, $\ell_1^{(k)}(x) = 0$. Set $\Omega_1 = \Omega$.
- If $x \in S_q$ for some $1 \leq q \leq Q$, then by Lemma 4.3.1 we have $b_1 = a_i$ for some $a_i \in A_q$. We have $\ell_1^{(q)}(x) = 1$ and, for all $k \neq q$, $\ell_1^{(k)}(x) = 0$. Set $\Omega_1^{(q)} = \{w^{(q)} \in \Omega^{(q)} \mid w_1^{(q)} = i\}$, and for all $k \neq q$ set $\Omega_1^{(k)} = \Omega^{(k)}$. Let $\Omega_1 = \prod_{k=1}^{Q} \Omega_1^{(k)}$.

Then in all cases, for each $1 \leq k \leq Q$ the set $\Omega_1^{(k)}$ is a cylinder set of length $\ell_1^{(k)}(x)$, where by a cylinder set of length 0 we mean the whole space $\Omega^{(k)}$. Furthermore, $b_1 = d_1(w, x)$ for all $w \in \Omega_1$. Now suppose we have obtained sets $\Omega_n \subseteq \cdots \subseteq \Omega_1$, so that for each $1 \leq k \leq Q$, $\Omega_n^{(k)}$ is a cylinder set of length $\ell_n^{(k)}(x)$, and that for all $1 \leq j \leq n$ and $w \in \Omega_j$ we have $b_j = d_j(w, x)$.

- If $x_{n+1} \in U_{a_i}$, then $\ell_{n+1}^{(k)}(x) = \ell_n^{(k)}(x)$ for all $1 \leq k \leq Q$, and for all $w \in \Omega_n$, $b_{n+1} = a_i = d_{n+1}(w, x)$. Therefore, set $\Omega_{n+1} = \Omega_n$.
- If $x_{n+1} \in S_q$, then $\ell_{n+1}^{(q)}(x) = \ell_n^{(q)}(x) + 1$ and, for all $k \neq q$, $\ell_{n+1}^{(k)}(x) = \ell_n^{(k)}(x)$. By Lemma 4.3.1, $b_{n+1} = a_i$ for some $a_i \in A_q$. Set $\Omega_{n+1}^{(q)} = \{w^{(q)} \in \Omega_n^{(q)} \mid w_{n+1}^{(q)} = i\}$, and for all $k \neq q$ let $\Omega_{n+1}^{(k)} = \Omega_n^{(k)}$. Set $\Omega_{n+1} = \prod_{j=1}^{Q} \Omega_n^{(j)}$. Then for each $w \in \Omega_{n+1}$ we have $b_{n+1} = d_{n+1}(w, x) = d_1(K^n(w, x))$.

The above shows that for each $1 \leq k \leq Q$, $\Omega_{n+1}^{(k)}$ is a cylinder set of length $\ell_{n+1}^{(k)}(x)$, and for all $w \in \Omega_{n+1}$ we have for all $1 \leq j \leq n+1$ that $b_j = d_j(w, x)$. If x is mapped to the switch region S_q infinitely many times, then $\ell_n^{(q)}(x) \to \infty$ and by the compactness of the cylinder sets, $\bigcap_{n=1}^{\infty} \Omega_n^{(q)}$ consists of one single point. If, on the other hand, x is mapped to S_q only finitely many times, then $\{\ell_n^{(q)}(x) \,|\, n \in \mathbb{N}\}$ is finite and $\bigcap_{n=1}^{\infty} \Omega_n^{(q)}$ is exactly a cylinder set. In both cases $\bigcap_{n=1}^{\infty} \Omega_n^{(q)}$ is non-empty, and since this holds for each q, the set $\bigcap_{n=1}^{\infty} \Omega_n$ also consists of at least one element. Furthermore, all elements $w \in \bigcap_{n=1}^{\infty} \Omega_n$ satisfy the requirement of the theorem. □

Remark 4.3.2. Notice that if an x hits every switch region infinitely many times, the above procedure leads to a unique w. Otherwise, one gets a cylinder set or even the whole space in case x has a unique expansion, i.e., in case the orbit of x only visits the uniqueness regions. More precisely, x has a unique β-expansion with digits in A if and only if for all $w \in \Omega$, $K^n(w,x) \notin \Omega \times \bigcup_{q=1}^{Q} S_q$ for all $n \geq 0$; or, equivalently, if $T^n x = L^n x$ for all $n \geq 0$.

4.4 Two sides of the same coin

In this section, we construct an isomorphism that links the random transformation K to the uniform Bernoulli shift. We will show that the set of points that have more than one expansion has full measure. Consider the probability space $(A^\omega, \mathcal{A}, P)$, where $\mathcal{A}$ is the product σ-algebra on A^ω, and P is the uniform product measure. In case $l_i = g_{i+j}$ for some i, j, we remove from A^ω all sequences that eventually end in $a_i a_m a_m \cdots$, and we call the new set D. Let $(D, \mathcal{D}, \mathbb{P})$ be the probability space obtained from $(A^\omega, \mathcal{A}, P)$ by restricting it to D, and let σ' be the left-shift on D. In this section, we let $\mathcal{B}$ denote the Borel σ-algebra on $\left[0, \frac{a_m}{\beta-1}\right]$. Let $\mathcal{F}$ be the product σ-algebra on Ω. Then $\mathcal{F} \times \mathcal{B}$ is the product σ-algebra on $\Omega \times \left[0, \frac{a_m}{\beta-1}\right]$. Define the function $\phi : \Omega \times \left[0, \frac{a_m}{\beta-1}\right] \to D$ by

$$\phi(w,x) = d_1(w,x) d_2(w,x) \cdots .$$

Then ϕ is measurable, $\phi \circ K = \sigma' \circ \phi$, and Theorem 4.3.1 states that ϕ is surjective. We will indicate a subset of $\Omega \times \left[0, \frac{a_m}{\beta-1}\right]$ on which ϕ is invertible. For $1 \leq q \leq Q$, let

$$Z_q = \left\{ (w,x) \in \Omega \times \left[0, \frac{a_m}{\beta-1}\right] \,\middle|\, \pi_2(K^n(w,x)) \in S_q \text{ i.o.} \right\},$$

$$D_q = \left\{ b_1 b_2 \cdots \in D \,\middle|\, \sum_{k=1}^{\infty} \frac{b_{j+k-1}}{\beta^k} \in S_q \text{ for infinitely many } j\text{'s} \right\},$$

and define the sets $Z = \bigcap_{q=1}^{Q} Z_q$ and $D^* = \bigcap_{q=1}^{Q} D_q$. It is clear that $\phi(Z) = D^*$, $K^{-1}(Z) = Z$ and $(\sigma')^{-1}(D^*) = D^*$. Let ϕ^* be the restriction of ϕ to Z. The next lemma states that ϕ^* is a bijection.

Lemma 4.4.1. *The map $\phi^* : Z \to D^*$ is a bimeasurable bijection.*

Proof. Let the sequence $b_1 b_2 \cdots$ be an element of D^*, and define for each $1 \leq q \leq Q$ the sequence $\{r_n^{(q)}\}_{n\geq 1}$ recursively:

$$\begin{aligned} r_1^{(q)} &= \min\left\{ j \geq 1 \,\Big|\, \sum_{k=1}^{\infty} \frac{b_{j+k-1}}{\beta^k} \in S_q \right\}, \\ r_n^{(q)} &= \min\left\{ j > r_{n-1}^{(q)} \,\Big|\, \sum_{k=1}^{\infty} \frac{b_{j+k-1}}{\beta^k} \in S_q \right\}. \end{aligned}$$

By Lemma 4.3.1, we know that $b_{r_n^{(q)}} = a_i \in A_q$. Set $w_n^{(q)} = i$. Take $w = (w^{(1)}, \ldots, w^{(Q)})$ and define $(\phi^*)^{-1} : D^* \to Z$ by

$$(\phi^*)^{-1}(b_1 b_2 \cdots) = \left(w, \sum_{k=1}^{\infty} \frac{b_k}{\beta^k} \right).$$

We can easily check that $(\phi^*)^{-1}$ is measurable and is the inverse of ϕ^*. □

Lemma 4.4.2. $\mathbb{P}(D^*) = 1$.

Proof. Fix q. We know that if the q-th switch region is given by $S_q = S_{a_i,\ldots,a_{i+j}}$, then it is bounded on the left by $\max\{g_{i+j}, l_i\}$ and it is bounded on the right by $\min\{g_{i+j+1}, l_{i+1}\}$. Let $x = \max\{g_{i+j}, l_i\}$ be the left endpoint of S_q, and suppose that the greedy expansion of x is given by $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$, with $b_k \in A$ for all $k \geq 1$. Notice that $b_k \neq a_m$ for infinitely many k's. Let

$$0 < \delta < \min\{g_{i+j+1} - x, l_{i+1} - x\}$$

and choose ℓ sufficiently large, such that

(i) $\sum_{k=\ell}^{\infty} \frac{a_m}{\beta^k} < \delta$ and

(ii) $b_\ell \neq a_m$.

Let $d_1 d_2 \cdots \in D$ be an arbitrary sequence of digits, and set

$$x_\ell = \sum_{k=1}^{\ell-1} \frac{b_k}{\beta^k} + \frac{a_m}{\beta^\ell} + \sum_{k=\ell+1}^{\infty} \frac{d_{k-\ell}}{\beta^k}.$$

Since, by the definition of the greedy expansion, b_ℓ is the largest element of A such that $\sum_{k=1}^{\ell} \frac{b_k}{\beta^k} \leq x$, we have

$$x_\ell \geq \sum_{k=1}^{\ell-1} \frac{b_k}{\beta^k} + \frac{a_m}{\beta^\ell} > x.$$

Also,

$$x_\ell \leq \sum_{k=1}^{\ell-1} \frac{b_k}{\beta^k} + \frac{a_m}{\beta^\ell} + \sum_{k=\ell+1}^{\infty} \frac{a_m}{\beta^k} < x + \delta < \min\{g_{i+j+1}, l_{i+1}\}.$$

So, $x_\ell \in S_q$. Define the set

$$D_q^* = \{d_1 d_2 \cdots \in A^\omega \mid d_k \cdots d_{k+\ell-1} = b_1 \cdots b_{\ell-1} a_m \text{ for infinitely many } k\text{'s}\}.$$

By the second Borel-Cantelli Lemma, $\mathbb{P}(D_q^*) = 1$ and, since obviously $D_q^* \subseteq D_q$, $\mathbb{P}(D_q) = 1$ also. This holds for all $1 \leq q \leq Q$, so we have $\mathbb{P}(D^*) = 1$. □

Using the product measure $\mathbb{P}$, we can define the K-invariant measure ν on $\mathcal{F} \times \mathcal{B}$ by setting

$$\nu(A) = \mathbb{P}(\phi(Z \cap A)).$$

As a direct consequence of Lemma 4.4.1 and Lemma 4.4.2, we have the following theorem.

Theorem 4.4.1. *Let $\beta > 1$, and suppose $A = \{0, a_1, \ldots, a_m\}$ is an allowable digit set. The dynamical systems $(\Omega \times \left[0, \frac{a_m}{\beta-1}\right], \mathcal{F} \times \mathcal{B}, \nu, K)$ and $(D, \mathcal{D}, \mathbb{P}, \sigma')$ are measurably isomorphic.*

As an immediate consequence of the isomorphism between the two dynamical systems, by Theorem 1.2.5 we have the following corollary about the entropy of K.

Corollary 4.4.1. *The entropy of the transformation K with respect to the measure ν is given by*

$$h_\nu(K) = \log(m+1).$$

Remark 4.4.1. Let $\mathcal{M}$ be the family of measures ρ defined on $(D, \mathcal{D})$, which are shift invariant and have $\rho(D^*) = 1$. Then for each $\rho \in \mathcal{M}$ we can define the measure ν_ρ on $(\Omega \times \left[0, \frac{a_m}{\beta-1}\right], \mathcal{F} \times \mathcal{B})$ by

$$\nu_\rho(A) = \rho(\phi(A \cap Z)).$$

It is clear that ν_ρ is K-invariant and that if $\nu_\rho \neq \mathbb{P}$, then

$$h_{\nu_\rho}(K) < \log(m+1).$$

Hence, ν is the only K-invariant measure with support Z and maximal entropy $\log(m+1)$.

We are interested in identifying the projection of the measure ν on the second coordinate, namely the measure $\nu \circ \pi_2^{-1}$ defined on $\left[0, \frac{a_m}{\beta-1}\right]$. To do so, we consider the purely discrete measures $\{\delta_k\}_{k \geq 1}$, defined on $\mathbb{R}$ as follows:

$$\delta_k(\{0\}) = \frac{1}{m+1}, \; \delta_k(\{a_1\beta^{-k}\}) = \frac{1}{m+1}, \ldots, \; \delta_k(\{a_m\beta^{-k}\}) = \frac{1}{m+1}.$$

Let δ be the corresponding infinite Bernoulli convolution,

$$\delta = \lim_{n\to\infty} \delta_1 * \cdots * \delta_n.$$

Proposition 4.4.1. $\nu \circ \pi_2^{-1} = \delta$.

Proof. Let $h : D \to \left[0, \frac{a_m}{\beta-1}\right]$ be given by $h(y) = \sum_{k=1}^{\infty} \frac{y_k}{\beta^k}$, where $y = y_1 y_2 \cdots$. Then, $\pi_2 = h \circ \phi$, and $\delta = \mathbb{P} \circ h^{-1}$. Since $\mathbb{P} = \nu \circ \phi^{-1}$, it follows that $\nu \circ \pi_2^{-1} = \delta$. $\square$

If $\beta \in (1,2)$ and $A = \{0,1\}$, then δ is an Erdös measure on $[0, \frac{1}{\beta-1}]$, and many things are already known. For example, if β is a Pisot number, i.e., a real algebraic integer bigger than 1 that has all its Galois conjugates in modulus less than one, then δ is singular with respect to Lebesgue measure (see [Erd39, Erd40, Sal63]). Furthermore, for almost all $\beta \in (1,2)$ the measure

δ is equivalent to Lebesgue measure (see [Sol95, MS98]). There are many generalizations of these results to the case of an arbitrary digit set (see [PSS00] for more references and results).

The transformation K described above is not the only one with these properties. There is another natural generalization of the transformation that was given in [DdV07] for the classical β-transformation. Instead of assigning a different die to each of the switch regions, we can also use the same $(m+1)$-sided die for each switch region and only consider the outcomes that are meaningful. We will define the dynamical system that generates the random β-expansions with arbitrary digits in this way. It can be shown that this system is isomorphic to the Bernoulli shift on A and hence also to the system described above.

Let the partition α be constructed as before. Let $\bar{\Omega} = \{0, \dots, m\}^{\omega}$, and let σ be the left-shift. Elements of $\bar{\Omega}$ now indicate a series of outcomes of our $(m+1)$-sided die and thus specify which transformation we choose. We first define the function $\bar{K} : \bar{\Omega} \times \left[0, \frac{a_m}{\beta-1}\right] \to \bar{\Omega} \times \left[0, \frac{a_m}{\beta-1}\right]$ by

$$\bar{K}(\bar{w}, x) = \begin{cases} (\bar{w}, \beta x - a_i), & \text{if } x \in U_{a_i}, \\ (\sigma(\bar{w}), \beta x - a_{\bar{w}_1}), & \text{if } x \in S_{a_i,\dots,a_{i+j}} \text{ and } \bar{w}_1 \in \{i, \dots, i+j\}, \\ (\sigma(\bar{w}), x), & \text{if } x \in S_{a_i,\dots,a_{i+j}} \text{ and } \bar{w}_1 \notin \{i, \dots, i+j\}. \end{cases}$$

Define the set S^* by

$$S^* = \bigcup_{i=0}^{m-1} \bigcup_{j=1}^{m-i} \left\{ \{\bar{w}\} \times S_{a_i,\dots,a_{i+j}} \mid \bar{w}_1 \notin \{i, \dots, i+j\} \right\}.$$

Let $X = \bar{\Omega} \times \left[0, \frac{a_m}{\beta-1}\right] \setminus S^*$ and $X^* = \bigcap_{n=0}^{\infty} \bar{K}^{-n}(X)$. Consider the restriction of $\bar{K}$ to this set X^* and call it $R = R_\beta$; that is,

$$R : X^* \to X^* : (\bar{w}, x) \mapsto \bar{K}(\bar{w}, x).$$

Notice that the set X^* only contains those combinations of $\bar{w}$'s and x's that yield 'valid' choices at every moment the die is thrown.

Let the dynamical system $(D, \mathcal{D}, \mathcal{P}, \sigma')$ be as before. Let $\bar{\mathcal{F}}$ be the product σ-algebra on $\bar{\Omega}$ and let $\mathcal{B}$ again be the Borel σ-algebra on $\left[0, \frac{a_m}{\beta-1}\right]$. Define

$$\mathcal{X} = \{A \cap X^* \mid A \in \bar{\mathcal{F}} \times \mathcal{B}\}.$$

Then, by defining the sequence of digits $(\bar{d}_n)_{n\geq 1}$ and the function $\bar{\phi}$ as we have done before, we can prove Theorem 4.3.1, Lemma 4.4.1 and Lemma 4.4.2 by only making slight adjustments to the proofs. If we also define the R-invariant measure $\bar{\nu}$ in a similar way, then we have shown that the system $(X^*, \mathcal{X}, \bar{\nu}, R)$ is isomorphic to $(D, \mathcal{D}, \mathbb{P}, \sigma')$.

Remark 4.4.2. We have established a measurable isomorphism between the dynamical systems $(\Omega \times \left[0, \frac{a_m}{\beta-1}\right], \mathcal{F} \times \mathcal{B}, \nu, K)$, $(X^*, \mathcal{X}, \bar{\nu}, R)$ and $(D, \mathcal{D}, \mathbb{P}, \sigma')$, but we still know very little about the measures ν and $\bar{\nu}$. An interesting question would be whether or not these measures are measures of maximal entropy, and if so, if they are the unique measures with this property. Another point of interest would be to find a measure whose projection onto the second coordinate is equivalent to Lebesgue measure.

Almost everything in Section 4.2 to Section 4.4 can be found in [DK07], except for Corollary 4.2.1, which is from [DK09b].

4.5 Size matters

From the previous, we see that in general for a $\beta > 1$ and an allowable digit set $A = \{0, a_1, \dots, a_m\}$ all elements in the interval $\left[0, \frac{a_m}{\beta-1}\right]$ have a β-expansion with digits in A and almost all elements have many different expansions. So, what happens if A is not allowable? Many articles have been written on this subject. For completeness we will state the most related results here. They are concerned with the size of the sets of numbers that have an expansion in a certain base β and a certain digit set. In other words, given a real number $\beta > 1$ and a finite set of real numbers $A = \{a_0, \dots, a_m\}$, what is the size of the set

$$C_\beta^A = \left\{ x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k} \,\middle|\, b_k \in A \right\}.$$

Size is usually expressed in Lebesgue measure, Hausdorff dimension and Hausdorff measure. These last two notions are given as follows. Let F be a subset of $\mathbb{R}^n$ and $\delta > 0$. If $\{E_k\}_{k\geq 1}$ is a countable family of subsets of $\mathbb{R}^n$ with diameter at most δ that cover F, then $\{E_k\}_{k\geq 1}$ is called a *δ-cover* of F. Let s be a non-negative number. Define

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{k=1}^{\infty} |E_k|^s \,\middle|\, \{E_k\} \text{ is a } \delta\text{-cover of } F \right\}.$$

Definition 4.5.1. Let $F \subset \mathbb{R}^n$. The *s-dimensional Hausdorff measure* of F is given by

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

The s-dimensional Hausdorff measure is translation invariant. There is a critical value of s, such that $\mathcal{H}^t(F) = \infty$, if $t < s$ and $\mathcal{H}^t(F) = 0$, if $t > s$. This critical value is the Hausdorff dimension of F.

Definition 4.5.2. Let $F \subseteq R^n$. Then the *Hausdorff dimension* of F is given by

$$\dim_H(F) = \inf\{s \geq 0 \,|\, \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 \,|\, \mathcal{H}^s(F) = \infty\}.$$

If $\dim_H(F) = s$, then $\mathcal{H}^s(F) \in [0, \infty]$ and this value is called the *Hausdorff measure* of F.

Another notion of size is the box-counting dimension, which is defined next.

Definition 4.5.3. Let $F \subset \mathbb{R}^n$ be non-empty and bounded and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The *lower* and *upper box-counting dimensions* of F are given by

$$\begin{aligned} \underline{\dim}_B(F) &= \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \\ \overline{\dim}_B(F) &= \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \end{aligned}$$

If these values are equal, then it is called the *box-counting dimension* of F and is denoted by $\dim_B(F)$.

For more information on the Hausdorff dimension and related concepts, the reader is referred to [Fal90] by Falconer, for example.

In 1995 Keane, Smorodinsky and Solomyak ([KSS95]) studied the size of sets of numbers having a β-expansion for the fixed digit set $\{0, 1, 3\}$. They proved that for $\beta < \frac{5}{2}$ every x in the interval $\left[0, \frac{3}{\beta-1}\right]$ has an expansion in base β and with digits in the set $\{0, 1, 3\}$. For $\beta > 3$ the set of numbers having such an expansion has Lebesgue measure zero. They called such expansions *expansions with deleted digits*. Moreover, in the same article they proved that there exists a sequence of algebraic integers β_k, all lying between $\frac{5}{2}$ and 3, such that the set of numbers that have an expansion in base β_k and digits in $\{0, 1, 3\}$ is small, i.e., has Hausdorff dimension less than 1.

At the same time, Pollicott and Simon studied in [PS95] the continuity of the Hausdorff dimension of the one parameter family of Cantor sets that is given by the above construction. They generalized the above situation to general digit sets of integers $S \subset \{0, 1, \ldots, (n-1)\}$. Let ℓ denote the cardinality of the set S. They proved that for Lebesgue almost all $\beta \in [\ell, n]$ the Hausdorff dimension is given by $\frac{\log \ell}{\log \beta}$. In contrast, they showed that under appropriate conditions on S, there is a dense set of $\beta \in [\ell, n]$ for which the Hausdorff dimension is strictly smaller than $\frac{\log \ell}{\log \beta}$.

In 1997 Lalley ([Lal97]) studied similar questions for Pisot β and digits in an arbitrary finite subset of $\mathbb{Z}[\beta]$. Recall that a real algebraic integer $\beta > 1$ is called Pisot if all its Galois conjugates are in modulus less than one. He gave an algorithm that computes the Hausdorff dimension of the set of numbers that has an expansion in a given Pisot base β and with a digit set contained in $\mathbb{Z}[\beta]$. In that same year Kenyon ([Ken97]) looked at the Hausdorff dimension of the set of numbers x that can be written as $x = \sum_{k=1}^{\infty} \frac{b_k}{3^k}$ with $b_k \in \{0, 1, u\}$ for some real u. He showed that for every irrational u, this set has Lebesgue measure 0. Moreover, he proved that for every rational $u = p/q$, the Hausdorff dimension of this set is less than 1, unless $p + q \equiv 0 \pmod 3$, in which case the set has a nonempty interior and Lebesgue measure $1/q$.

An overview and many results in this direction are given in [Sol98] by Solomyak. He considered general digit sets of real numbers with a real basis $\beta > 1$ and digit sets of complex numbers with a complex basis that has modulus bigger than one. He proved results about the Hausdorff dimension and

Hausdorff measure of the sets of numbers that have an expansion with such bases and digit sets. He also studied similar questions for transformations in $\mathbb{R}^d$ and many other things. Here we mentioned those results that relate to β-expansions with real bases and arbitrary real digits. Let A be a finite digit set of real numbers and set $\Gamma = A - A$. Consider the set of power series

$$B_\Gamma = \Big\{ f(x) = \sum_{j=0}^{\infty} f_j x^j \,\Big|\, f_j \in \Gamma \Big\}.$$

Define the set $\mathcal{M}_\Gamma$ to be the set of points $\beta > 1$, such that $1/\beta$ is a double zero of a power series from B_Γ, i.e.,

$$\mathcal{M}_\Gamma = \{\beta > 1 \,|\, \exists f \in \mathcal{B}_\Gamma,\ f \not\equiv 0,\ f(1/\beta) = f'(1/\beta) = 0\}.$$

The set C^A_β is said to satisfy the *transversality condition* at β if $\beta \notin \mathcal{M}_\Gamma$. One of many results from [Sol98] is the following theorem. For the proof, Solomyak refers to [PS95] by Pollicott and Simon for the first part and to [Sol95] for the second part.

Theorem 4.5.1 (Solomyak, [Sol98]). *Let $A = \{a_0, \ldots, a_m\} \subset \mathbb{R}$ be a finite set of real numbers. Then for λ a.e. $\beta \in \mathbb{R}_{>1} \backslash \mathcal{M}_\Gamma$, we have*

$$\begin{aligned} &\dim_H(C^A_\beta) = \tfrac{\log(m+1)}{\log \beta}, && \text{if } \beta > m+1, \\ &\lambda(C^A_\beta) > 0, && \text{if } \beta < m+1. \end{aligned}$$

As an illustration (and easy exercise), we study the Hausdorff dimension of the set of numbers that can be represented by an arbitrary set of three digits. As in Chapter 3, we will take this three digit set of the form $\{0, 1, u\}$ for some $u > 1$. This will be enough to cover all three digit sets, since other ones can be obtained just by scaling and translating. Theorem 4.5.1 from Solomyak implies for this situation that if we fix u, then for almost every $\beta > 1$ that is not in $\mathcal{M}_\Gamma$ we have:

- if $\beta > 3$, then $\dim_H(C^A_\beta) = \frac{\log 3}{\log \beta}$, and
- if $\beta < 3$, then $\lambda(C^A_\beta) > 0$.

What we will do is fix β instead and use results from [Fal87] by Falconer and [Mar54] by Marstrand to get similar results. We first need some definitions and notation. Let C be a closed subset of $\mathbb{R}^n$. A mapping $S : C \to C$ is called a *similarity* if there is a number $0 < c < 1$ such that $|S(x) - S(y)| = c|x - y|$ for all $x, y \in C$. The number c is called the *contraction ratio*. We call a finite family of similarities, $\{S_1, \ldots, S_k\}$, an *iterated function system* (IFS). A non-empty, compact subset F of C is an invariant set of the IFS if $F = \bigcup_{j=1}^k S_j(F)$. It is well-known that every IFS has a unique invariant set (see for example [Fal90]). This definition of an IFS can be generalized to a *graph-directed iterated function system*. We will not use them in this section, but the definition is given for future reference.

Definition 4.5.4. Let $\mathcal{N} = \{1, 2, \ldots, q\}$ be a set of nodes and $\mathcal{G}$ a set of directed edges starting and ending at a node from $\mathcal{N}$, such that $(\mathcal{N}, \mathcal{G})$ is a graph. For two nodes, $i, j \in \mathcal{N}$, let $\mathcal{G}_{i,j}$ denote the set of edges from node i to node j. Assume that for every two nodes, there is a (directed) path in the graph starting at one node and ending at the other. For each edge $e \in \mathcal{G}$, let $S_e : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity with contraction ratio $0 < c_e < 1$. Then the set of contractions $\{S_e \mid e \in \mathcal{G}\}$ is called a *graph-directed iterated function system*.

For a graph-directed iterated function system as in Definition 4.5.4 there is a unique family of non-empty compact sets $E_1, \ldots, E_q$ such that

$$E_i = \bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{G}_{i,j}} S_e(E_j). \tag{4.2}$$

The collection $\{E_1, \ldots, E_q\}$ is called a *family of graph-directed sets*. For more information on the graph-directed IFS, see [Fal97] by Falconer, for example.

An IFS, $\{S_1, \ldots, S_k\}$, is said to satisfy the *open set condition* if there is a non-empty, bounded, open set V, such that $\bigcup_{j=1}^{k} S_j V \subset V$, where the union is disjoint. Theorem 9.3 from [Fal90] tells us what the Hausdorff dimension is of the invariant set of an IFS that satisfies the open set condition. We state Theorem 9.3 here for convenience.

Theorem 4.5.2 (Falconer, [Fal90])**.** *Suppose that the open set condition holds for the family of similarities $\{S_1, \ldots, S_k\}$ with contraction ratios c_j, respectively. Let F be the invariant set of the IFS. Then $\dim_H(F) = \dim_B(F) = s$, where s is given by $\sum_{j=1}^{k} c_j^s = 1$. Moreover, for this value of s, the s-dimensional Hausdorff measure of the set F is strictly between 0 and ∞.*

If an IFS does not satisfy the open set condition, then we still have that $\dim_H(F) = \dim_B(F)$, but this value may be smaller than s.

Other important results in this respect are the projection theorems, Theorem I and Theorem II from [Mar54] by Marstrand. We also give them here, but state them in one theorem.

Theorem 4.5.3 (Marstrand, [Mar54])**.** *Any set of positive s-dimensional Hausdorff measure with $s > 1$ projects into a set of positive Lebesgue measure in almost all directions.*

Any set of positive s-dimensional Hausdorff measure with $s \leq 1$ projects into a set of Hausdorff dimension s in almost all directions.

For $\beta > 1$ and digit set $\{0, 1, u\}$, let C_β^u be the set of points that have an expansion in base β and with digits in $\{0, 1, u\}$. In what follows, we will keep β fixed and use a result from [Fal87] by Falconer to get the size of C_β^u. The next proposition just says that, if $\{0, 1, u\}$ is an allowable digit set for β, then C_β^u is the whole interval $\left[0, \frac{u}{\beta-1}\right]$.

Proposition 4.5.1. *Let $\beta, u > 1$. If $\beta \leq \min\{u + 1, \frac{2u-1}{u-1}\}$, then $C_\beta^u = \left[0, \frac{u}{\beta-1}\right]$. In particular, if $1 < \beta \leq 2$, then $C_\beta^u = \left[0, \frac{u}{\beta-1}\right]$ for all $u > 1$.*

Proof. The first part is just the condition that $\{0, 1, u\}$ is an allowable digit set for β. For the second part, it is enough to show that for $1 < \beta \leq 2$ the set $\{0, 1, u\}$ is allowable for all $u > 1$. Therefore we need to check that $1 \leq \frac{u}{\beta-1}$ and $u - 1 \leq \frac{u}{\beta-1}$. For the first inequality, notice that $\beta \leq 2 \leq u + 1$. For the second one we have $\frac{1}{\beta-1} \geq 1$, so $\frac{u}{\beta-1} \geq u > u - 1$. $\square$

Recall the definition of the greedy and lazy partition points, Definition 4.2.2. Consider the intervals $I_1 = (g_1, l_1) = \left(\frac{1}{\beta}, \frac{u}{\beta(\beta-1)}\right)$ and $I_2 = (g_2, l_2) = \left(\frac{u}{\beta}, \frac{u}{\beta(\beta-1)} + \frac{1}{\beta}\right)$. These intervals are both non-empty, if and only if $\{0, 1, u\}$ is allowable for β. If one of the intervals I_1 or I_2 is empty, not every x in the interval $\left[0, \frac{u}{\beta-1}\right]$ has an expansion of the form (2.3). This means that C^u_β is no longer the complete interval. We would like to give the Lebesgue measure and Hausdorff dimension of the set C^u_β in case at least one of the sets I_1, I_2 is empty. First suppose that $\beta > \max\{u + 1, \frac{2u-1}{u-1}\}$. This implies that $\beta > 3$. In this case both I_1 and I_2 are empty. The following proposition gives the Hausdorff dimension of C^u_β in this case.

Proposition 4.5.2. *Let $u > 1$ and $\beta > \max\left\{u + 1, \frac{2u-1}{u-1}\right\}$. Then*

$$\dim_H(C^u_\beta) = \frac{\log 3}{\log \beta}.$$

Proof. Define the similarities S_1, S_2, S_3 from $\left[0, \frac{u}{\beta-1}\right]$ to itself by $S_1 x = \frac{1}{\beta}x$, $S_2 x = \frac{1}{\beta}x + \frac{1}{\beta}$ and $S_3 x = \frac{1}{\beta}x + \frac{u}{\beta}$. Let the transformation S, defined on the set of non-empty, compact subsets of $\left[0, \frac{u}{\beta-1}\right]$, be given by $S(E) = \bigcup_{j=1}^{3} S_j(E)$. Then

$$C^u_\beta = \bigcap_{k=0}^{\infty} S^k\left(\left[0, \frac{u}{\beta-1}\right]\right),$$

i.e., C^u_β is the unique attractor of the system of similarities $\{S_1, S_2, S_3\}$. Notice that for the open, bounded interval $\left(0, \frac{u}{\beta-1}\right)$ we have

$$\bigcup_{j=1}^{3} S_j\left(\left(0, \frac{u}{\beta-1}\right)\right) \subset \left(0, \frac{u}{\beta-1}\right),$$

where the union on the left-hand side is disjoint. This means that $\left(0, \frac{u}{\beta-1}\right)$ satisfies the open set condition. By Theorem 4.5.2 we have that the Hausdorff dimension of C^u_β is given by the value s for which $3 \cdot \frac{1}{\beta^s} = 1$. So,

$$\dim_H(C^u_\beta) = \frac{\log 3}{\log \beta}. \qquad \square$$

Theorem 4.5.2 gives us even more. It tells us that in this case the box-counting dimension of C^u_β is equal to its Hausdorff dimension and that the Hausdorff measure of C^u_β is strictly between 0 and ∞. We have the following picture.

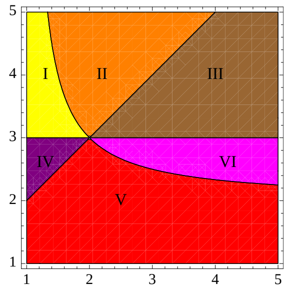


Figure 4.6 The areas given by the restrictions on β and u. On the vertical axis we see β and on the horizontal axis u.

In area V the digit set $\{0, 1, u\}$ is allowable and in area II we have the open set condition. In area I and IV we have $u + 1 < \beta < \frac{2u-1}{u-1}$ and areas III and VI are given by $\frac{2u-1}{u-1} < \beta < u + 1$. The next lemma gives that for $\beta > 3$, the Lebesgue measure of the set C^u_β is zero.

Lemma 4.5.1. *Suppose* $\beta > 3$. *Then* $\lambda(C^u_\beta) = 0$.

Proof. Let $\beta > 3$ and suppose $\lambda(C^u_\beta) > 0$. It follows from the definition of C^u_β, that

$$\beta C^u_\beta = C^u_\beta \cup (C^u_\beta + 1) \cup (C^u_\beta + u). \tag{4.3}$$

Thus,

$$3\lambda(C^u_\beta) \geq \lambda(C^u_\beta \cup (C^u_\beta + 1) \cup (C^u_\beta + u)) = \lambda(\beta C^u_\beta) = \beta\lambda(C^u_\beta),$$

which gives $\lambda(C^u_\beta) = 0$. □

We are interested in determining the Hausdorff dimension of the set C^u_β. Theorem 1 in [Fal87] gives this dimension for $\beta > 3$ and Lebesgue almost all u. Using the technique from this proof, we can get the Hausdorff dimension for all $\beta > 2$ and Lebesgue almost all u.

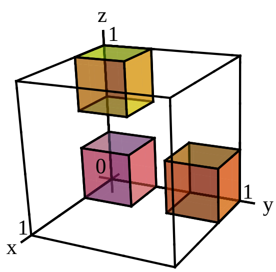


Figure 4.7 The sets $S_j(J)$ for $\beta = e$ and $u = \pi$.

Theorem 4.5.4 (cf. Falconer ([Fal87])). *Let $\beta > 2$. Then for almost all u, the set C_β^u has Hausdorff dimension* $\min\left\{1, \frac{\log 3}{\log \beta}\right\}$.

Proof. Let $\beta > 2$ be given. Define the vectors $\mathbf{c}_j \in \mathbb{R}^2$, $j \in \{1,2,3\}$, by

$$\mathbf{c}_1 = (0,0),\ \mathbf{c}_2 = (1 - 1/\beta, 0),\ \mathbf{c}_3 = (0, 1 - 1/\beta).$$

Note that the matrix

$$C = \begin{pmatrix} \mathbf{c}_{1,1} & \mathbf{c}_{1,2} & 1 - 1/\beta \\ \mathbf{c}_{2,1} & \mathbf{c}_{2,2} & 1 - 1/\beta \\ \mathbf{c}_{3,1} & \mathbf{c}_{3,2} & 1 - 1/\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 - 1/\beta \\ 1 - 1/\beta & 0 & 1 - 1/\beta \\ 0 & 1 - 1/\beta & 1 - 1/\beta \end{pmatrix}$$

is invertible. Therefore, we can always find a vector $(t_1, t_2, t_3) \in \mathbb{R}^3$ such that $\frac{1}{\beta} \cdot (0, 1, u)^t = C(t_1, t_2, t_3)^t$. The idea of the proof is to construct a set in $\mathbb{R}^3$ that is the attractor of three similarities with contraction ratio $1/\beta$ which satisfy the open set condition. This immediately gives that the Hausdorff dimension of this attractor is equal to $\log 3/\log \beta$. We then show that for all u there is a projection of this attractor to the set C_β^u. Theorem 4.5.3 then gives us the desired result.

We first construct the similarities in $\mathbb{R}^3$. For $j \in \{1,2,3\}$, let the similarities $S_j : \mathbb{R} \to \mathbb{R}^3$ be given by

$$S_j(x, y, z) = \left(\frac{1}{\beta}x, \frac{1}{\beta}y + \mathbf{c}_{j,1}, \frac{1}{\beta}z + \mathbf{c}_{j,2}\right).$$

Let $J = (0,1)^3$. Since $\beta > 2$, we have $1/\beta < 1 - 1/\beta$. This implies that the hypercubes $S_j(J)$ are mutually disjoint and contained in J. See Figure 4.7. So, the mappings S_j satisfy the open set condition. Hence, there is a unique non-empty, compact set $F \subset \mathbb{R}^3$, such that $F = \bigcup_{j=1}^3 S_j(F)$. Moreover, $\dim_H(F) = \log 3/\log \beta$.

For each $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, we can define the mapping $U_\mathbf{t} : \mathbb{R}^3 \to \mathbb{R}$ by

$$U_\mathbf{t}(x, y, z) = x + t_1 y + t_2 z.$$

Let $T^* : \mathbb{R} \to \mathbb{R}$ be given by $T^* x = x/\beta$. Then

$$U_\mathbf{t} \circ S_j = (T^* \circ U_\mathbf{t}) + \mathbf{c}_j \cdot \mathbf{t},$$

where the dot denotes the inner product. Thus

$$U_\mathbf{t}(F) = \bigcup_{j=1}^{3} (U_\mathbf{t} \circ S_j)(F) = \bigcup_{j=1}^{3} T^*(U_\mathbf{t}(F)) + \mathbf{c}_j \cdot \mathbf{t}.$$

This implies that $U_\mathbf{t}(F)$ is the unique, non-empty, compact, invariant set associated with the mappings $T^* + \mathbf{c}_j \cdot \mathbf{t}$. By Theorem 4.5.3, $\dim_H U_\mathbf{t}(F) = \min\{1, \log 3/\log \beta\}$ for almost all $\mathbf{t} \in \mathbb{R}^2$. For each $t_3 \in \mathbb{R}$, we have

$$(U_\mathbf{t}(F) + t_3) = \bigcup_{j=1}^{3} T^*(U_\mathbf{t}(F) + t_3) + \mathbf{c}_{j,1}t_1 + \mathbf{c}_{j,2}t_2 + (1 - 1/\beta)t_3.$$

So $U_\mathbf{t}(F) + t_3$ is the unique attractor of the similarities $T^* + \mathbf{c}_{j,1}t_1 + \mathbf{c}_{j,2}t_2 + (1 - 1/\beta)t_3$. The Hausdorff dimension of a set is invariant under translation, so $\dim_H(U_\mathbf{t}(F) + t_3) = \min\{1, \log 3/\log \beta\}$ for almost all vectors (t_1, t_2, t_3). Since for every u there is a vector (t_1, t_2, t_3) such that $U_\mathbf{t}(F) + t_3 = C_\beta^u$, we have the theorem. □

Theorem 4.5.4 completes the description of the areas in Figure 4.6. It implies that in areas I and III, the Hausdorff dimension of C_β^u is almost everywhere equal to $\log 3/\log\beta$ and in areas IV and VI the Hausdorff dimension is almost everywhere equal to 1. These results are summarized in the next table.

area	β	I_1, I_2	size
I	$\beta > 3$	$I_1 = \emptyset$, $I_2 \neq \emptyset$	$\forall\beta$, a.e. u, $\dim_H(C_\beta^u) = \frac{\log 3}{\log\beta}$
II	$\beta > 3$	I_1, $I_2 = \emptyset$	$\forall\beta$, u, $\dim_H(C_\beta^u) = \frac{\log 3}{\log\beta}$
III	$\beta > 3$	$I_1 \neq \emptyset$, $I_2 = \emptyset$	$\forall\beta$, a.e. u, $\dim_H(C_\beta^u) = \frac{\log 3}{\log\beta}$
IV	$\beta \leq 3$	$I_1 = \emptyset$, $I_2 \neq \emptyset$	$\forall\beta$, a.e. u, $\lambda(C_\beta^u) > 0$
V	$\beta \leq 3$	I_1, $I_2 \neq \emptyset$	$\forall\beta$, u, $C_\beta^u = [0, u/(\beta-1)]$
VI	$\beta \leq 3$	$I_1 \neq \emptyset$, $I_2 = \emptyset$	$\forall\beta$, a.e. u, $\lambda(C_\beta^u) > 0$

CHAPTER 5

TILE IT RIGHT

5.1 How Pisot can you go?

Much of the literature about β-expansions with digits in $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is concerned with bases, that have nice algebraic properties, the Pisot β's. We have already encountered them before, but we will give a formal definition here, together with the definition of Salem numbers.

Definition 5.1.1. Let $\beta \in \mathbb{R}_{>1}$ be an algebraic integer with minimal polynomial $P(x) = x^d - c_1 x^{d-1} - \cdots - c_d$. Denote the other roots of $P(x)$, or Galois conjugates of β, by $\beta_2, \ldots, \beta_d$. Then β is called a *Pisot number* if $|\beta_j| < 1$ for all $2 \le j \le d$. The number β is called a *Salem number* if $|\beta_j| \le 1$ for all $2 \le j \le d$.

The most famous Pisot number is probably the golden mean, but all n-bonacci numbers, i.e., the real solutions bigger than 1 of the equations $x^n - x^{n-1} - \cdots - 1 = 0$, are Pisot numbers as well. For $n = 3$, this number is called the Tribonacci number. The smallest Pisot number is the real solution of the equation $x^3 - x - 1 = 0$. If β is a Pisot number, much more can be said about the properties of the β-expansions. We will state here, what we will need in the following sections. The theorem below can be found in [Sch80] by Schmidt.

Theorem 5.1.1 (Schmidt, [Sch80])**.** *Let $\beta > 1$ be a real number and T_β the greedy β-transformation with digit set $\{0, 1, \ldots, \lfloor\beta\rfloor\}$, defined on the interval* $[0, 1)$*. Let* $\mathrm{Per}(\beta)$ *denote the set of points from* $[0, 1)$ *of which the expansion generated by T_β is eventually periodic. Then the following hold.*

(i) If $\mathbb{Q} \cap [0, 1) \subseteq \mathrm{Per}(\beta)$, then β is either a Pisot or a Salem number.

(ii) If β is a Pisot number, then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$.

This chapter is about multiple tilings associated to transformations that generate β-expansions. Before we explain how to obtain such tilings, we give the necessary definitions. We consider multiple tilings by translation. They are given as follows. For $r > 0$ we use $B(\mathbf{y}, r)$ to denote the open ball with center $\mathbf{y}$ and radius r.

Definition 5.1.2. Let H be a Euclidean space and $\{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$ a finite collection of compact subsets of H. Suppose that for each set $\mathcal{D}_k$, there is a set of translation vectors $\tau_k \subseteq H$. Let

$$\mathcal{T} = \{\mathcal{D}_k + \mathbf{t}_k \,|\, \mathbf{t}_k \in \tau_k,\ 1 \leq k \leq n\}.$$

Then $\mathcal{T}$ is called a *multiple tiling* of H if there exists an $\textsc{m} \geq 1$, such that all of the following hold.

(mt1) Each set $\mathcal{D}_k$ is the closure of its interior.

(mt2) The family $\mathcal{T}$ is *locally finite*, i.e., for all $\mathbf{y} \in H$ there is a positive r such that the set $\{\mathbf{t}_k \in \tau_k \,|\, (\mathcal{D}_k + \mathbf{t}_k) \cap B(\mathbf{y}, r) \neq \emptyset,\ 1 \leq k \leq n\}$ is finite.

(mt3) The space H is covered by elements from $\mathcal{T}$, i.e.,

$$H = \bigcup_{k=1}^{n} \bigcup_{\mathbf{t}_k \in \tau_k} (\mathcal{D}_k + \mathbf{t}_k).$$

(mt4) Lebesgue almost all $\mathbf{y} \in H$ lie in exactly $\textsc{m}$ elements of $\mathcal{T}$.

The number $\textsc{m}$ is called the *covering degree* of the multiple tiling. The sets $\mathcal{D}_k$ are called the *prototiles*. A *tiling* is a multiple tiling with covering degree 1.

To show that a certain construction gives a multiple tiling, we usually show some properties of the set of translation vectors. We want this set to be big enough, but not too big. The first property we give here is relatively denseness, which says that the set is big enough.

Definition 5.1.3. A subset $E \subset \mathbb{R}^n$ is called *relatively dense* if there is an $R > 0$, such that for each $\mathbf{y} \in \mathbb{R}^n$, the set $B(\mathbf{y}, R) \cap E$ is not empty.

The next property says that the set is not too big.

Definition 5.1.4. A subset $E \subset \mathbb{R}^n$ is called *uniformly discrete* if there is an $r > 0$, such that for each $\mathbf{x} \in E$ the set $B(\mathbf{x}, r) \cap E$ contains only one element.

A set that is both relatively dense and uniformly discrete is called a *Delone set*. A subgroup of a locally compact abelian group is a *lattice* if it is a Delone set.

A way to show these properties for a set $E \subset \mathbb{R}^n$ is by showing that the set is a model set for a cut and project scheme, as defined in [Moo97] by Moody and originally in [Mey72] by Meyer. In [Moo97] Moody studied, among other things, sets which are Delone sets. In this article, Moody gave a detailed exposition of Meyer's theory, which was developed in [Mey72]. According to Meyer, model sets for cut and project schemes are Delone, a proof can be found in [Moo97]. We give the definitions here.

Definition 5.1.5. Let $n \geq 1$. Consider the direct product $\mathbb{R}^n \times K$, where K is a locally compact abelian group and let D be a lattice in $\mathbb{R}^n \times K$. Let $\pi_1 : \mathbb{R}^n \times K \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times K \to K$ be the canonical projections. The pair $(\mathbb{R}^n \times K, D)$ is called a *cut and project scheme* if the projections satisfy

- $\pi_1 : D \to \mathbb{R}^n$ is injective, and
- $\pi_2(D)$ is dense in K.

A subset $\Lambda \subset \mathbb{R}^n$ is called a *model set* for $(\mathbb{R}^n \times K, D)$, if there is a set $E \subset K$ with compact closure and non-empty interior, such that

$$\Lambda = \{\pi_1(d) \,|\, d \in D,\ \pi_2(d) \in E\}.$$

Theorem 5.1.2 (Moody, [Moo97]). *Any model set is a Delone set.*

A construction of tiles associated to T_β was first given in 1989 by Thurston in [Thu89]. For β equal to the Tribonacci number, Rauzy already studied the tiles in [Rau82]. Many people from many different fields of mathematics have since then studied the construction. An overview of results in this direction can be found in the survey paper [BS05] by Berthé and Siegel. The construction of the tiles associated to the transformation T_β, as it is given there, is as follows.

Let β be a Pisot number and T_β the greedy β-transformation with digit set $\{0, 1, \ldots, \lfloor\beta\rfloor\}$. Suppose that the real Galois conjugates of β are given by $\beta_2, \ldots, \beta_r$ and that the complex Galois conjugates are $\beta_{r+1}, \overline{\beta_{r+1}}, \ldots, \beta_{r+s}, \overline{\beta_{r+s}}$. So, $r + 2s = d$. Set $\beta_1 = \beta$. Let $\Gamma_j : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j)$, $1 \leq j \leq d$, be the isomorphism defined by $\Gamma_j(\beta) = \beta_j$ and $\Gamma_j(q) = q$ for $q \in \mathbb{Q}$. Let $\mathbb{K}_\beta = \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}$. Define the function $\Phi_\beta : \mathbb{Q}(\beta) \to \mathbb{K}_\beta$ by

$$\Phi_\beta(x) = (\Gamma_2(x), \ldots, \Gamma_r(x), \Gamma_{r+1}(x), \ldots, \Gamma_{r+s}(x)).$$

For $x \in [0, 1)$, denote the β-expansion of x generated by T_β, by $d(x)$. Note that if β is a Pisot number, then Theorem 5.1.1 implies that $d(\beta - \lfloor\beta\rfloor)$ is eventually periodic. Let $\bar{\mathcal{S}}_\beta$ be the β-shift, i.e., the closure of the set of two-sided sequences of which each left truncated sequence is a digit sequence produced by T_β:

$$\bar{\mathcal{S}}_\beta = \overline{\{u = (u_k)_{k\in\mathbb{Z}} \,|\, \forall k \in \mathbb{Z}\, \exists x \in [0, 1)\, :\, u_k u_{k+1} \cdots = d(x)\}}.$$

By Theorem 2.1.1, we also have

$$\bar{\mathcal{S}}_\beta = \{u = (u_k)_{k\in\mathbb{Z}} \,|\, \forall k \in \mathbb{Z}\, u_k u_{k+1} \cdots \preceq d^*(1)\},$$

with $d^*(1)$ as given before Theorem 2.1.1. This set is compact in the product topology. Let $\bar{\mathcal{S}}^l_\beta$ be the set of left-sided sequences $w = (w_k)_{k\leq 0}$, such that there is a right-sided sequence $u = (u_k)_{k\geq 1}$ with $w \cdot u \in \bar{\mathcal{S}}_\beta$, where $w \cdot u$ is defined as in Section 1.2.2. Define the function $\varphi_\beta : \bar{\mathcal{S}}^l_\beta \to \mathbb{K}_\beta$ by

$$\begin{aligned}
\varphi_\beta(w) &= \lim_{n\to\infty} \Phi_\beta\left(\sum_{k=0}^{n} w_{-k}\beta^k\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=0}^{n} w_{-k}\beta_2^k, \ldots, \sum_{k=0}^{n} w_{-k}\beta_r^k, \sum_{k=0}^{n} w_{-k}\beta_{r+1}^k, \ldots, \sum_{k=0}^{n} w_{-k}\beta_{r+s}^k\right).
\end{aligned}$$

Many tiling properties are related to a specific set, called the *central tile*. It is given by

$$\mathcal{T}_c^\beta = \varphi_\beta(\bar{\mathcal{S}}_\beta^l).$$

For each $x \in \mathbb{Z}[\beta] \cap [0,1)$, define the tile $\mathcal{T}_x^\beta$ by

$$\mathcal{T}_x^\beta = \Phi_\beta(x) + \varphi_\beta(\{w \in \bar{\mathcal{S}}_\beta^l \mid w \cdot d(x) \in \bar{\mathcal{S}}_\beta\}).$$

Consider the family $\mathcal{T}^\beta = \{\mathcal{T}_x^\beta\}_{x \in \mathbb{Z}[\beta]\cap[0,1)}$. We list some of the properties of this family below.

The definition of classical β-expansions can easily be extended from numbers in $[0,1)$ to numbers in $[0,\infty)$ in the following way. Suppose that $x \in \left[\frac{1}{\beta^n}, \frac{1}{\beta^{n+1}}\right)$ for some $n \geq 0$. Then $\frac{x}{\beta^{n+1}} \in [0,1)$. Let $d\left(\frac{x}{\beta^{n+1}}\right) = d_1 d_2 \cdots$ be the digit sequence of $\frac{x}{\beta^{n+1}}$ generated by T_β. Then the expression

$$x = d_1\beta^n + \cdots + d_n\beta + d_{n+1} + \sum_{k=1}^{\infty} \frac{d_{k+n+1}}{\beta^k}$$

is the *classical β-expansion of x generated by T_β*. We say that a point $x \in [0,\infty)$ has a *finite expansion* if the corresponding digit sequence ends in an infinite string of 0's. The set of points in $[0,\infty)$ with a finite expansion is usually denoted by Fin(β). If the set of numbers that has such a finite expansion is exactly $\mathbb{Z}[1/\beta]_{\geq 0}$, then β is said to satisfy the *finiteness property* (F):

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta]_{\geq 0}. \tag{5.1}$$

This property was first introduced in [FS92] by Frougny and Solomyak. In 1999 Akiyama ([Aki99]) and Praggastis ([Pra99]), independently of one another, gave fundamental properties of the tiles in $\mathcal{T}^\beta$. They both showed, for example, that if β is a Pisot unit that satisfies the (F) property, then the origin is an inner point of the central tile. In [Aki99], Akiyama showed that this implies that each tile is the closure of its interior and that the boundary of each tile is nowhere dense. He also proved the following result.

Theorem 5.1.3 (Akiyama, [Aki99])**.** *If β is a Pisot number, then the set $\Xi(\mathbb{Z}[\beta]_{>0})$ is dense in $\mathbb{K}_\beta$.*

In [Aki02] Akiyama proved many things about the construction. He showed that there are only finitely many different sets

$$\varphi_\beta(\{w \in \bar{\mathcal{S}}_\beta^l \mid w \cdot d(x) \in \bar{\mathcal{S}}_\beta\}).$$

This makes them suitable as prototiles. In [Aki02] we can find the following result about points with a purely periodic expansion.

Theorem 5.1.4 (Akiyama, [Aki02])**.** *The number of elements from $\mathbb{Z}[\beta]_{\geq 0}$ of which the β-expansion generated by T_β is purely periodic, is finite.*

Akiyama also proved in [Aki02] that the family $\mathcal{T}^\beta$ is locally finite. Combined results from [Aki99] and [Aki02] give that the tiles from Thurston's construction give a tiling of $\mathbb{K}_\beta$ in case β is a Pisot unit, satisfying the (F) property. The goal in [Aki02] was to prove properties of the above construction for Pisot units that satisfy a weaker version of the (F) property, called the (W) property. In [Aki02] Akiyama stated it as follows.

$$\forall x \in \mathbb{Z}[1/\beta]_{\geq 0}\, \forall \epsilon > 0\, \exists y, z \in \text{ Fin}(\beta) \text{ with } |z| < \epsilon \text{ and } x = y - z. \tag{5.2}$$

He proved the following theorem.

Theorem 5.1.5 (Akiyama, [Aki02])**.** *If β is a Pisot unit that satisfies the (W) property, then the $(d-1)$-dimensional Lebesgue measure of the boundary of the tiles, $\mathcal{T}_x^\beta$, $x \in \mathbb{Z}[\beta] \cap [0,1)$, is zero.*

In [Aki02], it is shown that the (W) property is equivalent to the fact that the central tile has an *exclusive inner point*, i.e., a point that lies in exactly one tile and is an inner point of that tile. Let $\mathcal{P}_\beta$ denote the set of points in $\mathbb{Z}[\beta] \cap [0,1)$ of which the expansion generated by T_β is purely periodic. Akiyama proved that the origin is an element of a tile $\mathcal{T}_x^\beta$, if and only if $x \in \mathcal{P}_\beta$. He showed that the origin is an inner point of the union of those tiles, that correspond to points with a purely periodic expansion. He proved the following result.

Theorem 5.1.6 (Akiyama, [Aki02])**.** *The central tile contains an exclusive inner point if and only if β satisfies the (W) property.*

In [Aki02], Akiyama gave a weaker version of the (W) property, only involving the set $\mathcal{P}_\beta$. He called this property the (W') property. It is given as follows.

$$\forall x \in \mathcal{P}_\beta\, \forall \epsilon > 0\, \exists y, z \in \text{ Fin}(\beta) \text{ with } |z| < \epsilon \text{ and } x = y - z. \tag{5.3}$$

He proved that the (W) property is equivalent to the (W') property. Hence, it is enough to check this property for the points with a purely periodic expansion.

The question of whether or not the construction gives a tiling when conditions are relaxed, is equivalent to a number of questions in different fields in mathematics and computer science, like spectral theory (see [Sie04] by Siegel), the theory of quasicrystals (see [ABEI01] by Arnoux et al.), discrete geometry (Ito and Rao, [IR06]) and automata ([Sie04]). As said before, in [Aki02] Akiyama replaced the condition (F) by the weaker condition (W). Akiyama also stated there that it is likely that all Pisot units satisfy this condition (W). It is conjectured that we get a tiling of the appropriate Euclidean space for all Pisot β. This is known as the Pisot conjecture and it is discussed at length in the survey paper [BS05] by Berthé and Siegel.

Conjecture 5.1.1 (Pisot Conjecture)**.** *For every Pisot number β, the family $\mathcal{T}^\beta$ as constructed above gives a tiling of the appropriate Euclidean space.*

The construction of the tiling makes use of the set of two-sided "digit sequences". In some sense, this construction makes the non-invertible transformation T_β invertible at the level of sequences. Another construction with this property is the natural extension. Thus, this tiling and the natural extension of T_β are closely related concepts.

In this chapter, we will consider a general class of piecewise linear, expanding maps with constant slope β, that can be used to generate β-expansions with arbitrary digits. Most of the transformations that we encountered in previous chapters fall into this class, but there are many other examples. We will introduce some of these examples in this chapter. We will construct a multiple tiling for such transformations under certain conditions on β and the digit set. Since this construction uses the β-expansions generated by the transformation, we first need a characterization of this set of sequences. This is done in Section 5.2, where we first introduce the class of transformations. In this section we put very mild restrictions on β and the digit set. In Section 5.3 we introduce the set-up and conditions that we need for the multiple tiling. We make a natural extension for the transformations for which β is a Pisot unit and the digit set A is a finite subset of $\mathbb{Q}(\beta)$. We discuss the shape of the natural extension domain and introduce a collection of sets. This collection is the set of prototiles for the multiple tiling that we will construct later. In Section 5.3 we already give a condition under which the collection of prototiles is finite. In Section 5.4 we introduce the collection of tiles and show that it gives a multiple tiling. We give conditions under which we have in fact a tiling and show that the tiles give a tiling if and only if the natural extension domain gives a tiling of the torus. We will use ideas of proofs from the articles [Aki99], [Aki02] and [IR06]. This last article is on tilings associated to substitutions. This is a closely related subject, but it goes too far to give the details here. Therefore, we will give the references when needed. In the last part of this chapter, we give examples of tilings and of multiple tilings with covering degree bigger than 1. We consider a particular multiple tiling, obtained from the symmetric β-transformation, for β equal to the Tribonacci number, in which almost every point is contained in two different tiles. Since the Tribonacci number is a Pisot unit, this gives a negative answer to the Pisot conjecture in our setting. So, for the more general class of transformations that we consider in this chapter, the Pisot conjecture does not hold.

5.2 Keep it lexicographical

The class of piecewise linear, expanding transformations with constant slope that we will be considering is the following.

Let I be a finite ordered index set and X a disjoint union of non-empty sets $X_i \subset \mathbb{R}$, $i \in I$. Let $A = \{a_i \in \mathbb{R} \,|\, i \in I\}$ and define the transformation $T : X \to X$ by $Tx = \beta x - a_i$ for all $x \in X_i$, $i \in I$. Assume that X is a bounded set and that $\sup X_i \leq \inf X_j$ if $i < j$, $i, j \in I$. The β-transformations that we have encountered in previous chapters, can all be obtained if we choose $I = A$ and $a_i = i$ for all $i \in I$, see Example 5.2.1 and Example 5.2.2. As before, let $\prec$ denote the lexicographical ordering. We are interested in the β-expansions with digits in A, generated by the transformation T. Note that these expansions are characterized by sequences in I^ω. Therefore, we define the digit sequences generated by T to be sequences in I^ω. Since the transformations from previous chapters are obtained by taking $I = A$, this does not contradict earlier definitions.

Definition 5.2.1. Let T be as in the preceding paragraph. For $x \in X$, the sequence $b(x) = b_1(x)b_2(x)\cdots \in I^\omega$ satisfying $b_k(x) = i$ if $T^{k-1}x \in X_i$, $i \in I$, for each $k \geq 1$, is called the *digit sequence of x generated by T*. A sequence $u \in I^\omega$ is called *T-admissible* if $u = b(x)$ for some $x \in X$.

We also redefine the value of a sequence.

Definition 5.2.2. The *value* of a sequence $u = (u_k)_{k\geq 1} \in I^\omega$ is $\sum_{k=1}^{\infty} \frac{a_{u_k}}{\beta^k}$ and is denoted by $\boldsymbol{.}u$.

Hence, we still have $x = \boldsymbol{.}b(x)$. Note that $Tx = \beta x - a_{b_1(x)}$, so for each $n \geq 1$ we have

$$x = \frac{a_{b_1(x)}}{\beta} + \frac{Tx}{\beta} = \cdots = \sum_{k=1}^{n} \frac{a_{b_k(x)}}{\beta^k} + \frac{T^n x}{\beta^n}. \tag{5.4}$$

Since X is bounded, we have $\lim_{n\to\infty} \frac{T^n x}{\beta^n} = 0$ and thus $x = \sum_{k=1}^{\infty} \frac{a_{b_k(x)}}{\beta^k}$. The aim of this section is to describe the set of T-admissible sequences, under some conditions on T. Theorem 5.2.1 gives a characterization of these sequences. We need the following lemma, which is a generalization of Proposition 2.2.3.

Lemma 5.2.1. *Let $x, y \in X$. Then $x < y$ is equivalent to $b(x) \prec b(y)$.*

Proof. Clearly, $b(x) = b(y)$ is equivalent to $x = y$. So we can assume that there exists some $k \geq 1$ such that $b_1(x)\cdots b_{k-1}(x) = b_1(y)\cdots b_{k-1}(y)$ and $b_k(x) \neq b_k(y)$. Then $x < y$ is equivalent to $T^{k-1}x < T^{k-1}y$ by (5.4). Since we have $T^{k-1}x \in X_{b_k(x)}$, $T^{k-1}y \in X_{b_k(y)}$, we obtain that $T^{k-1}x < T^{k-1}y$, $b_k(x) \neq b_k(y)$, is equivalent to $b_k(x) < b_k(y)$. This proves the lemma. $\square$

From now on, we assume that all sets X_i are half-open intervals.

Definition 5.2.3. Let $\beta > 1$, I be a finite ordered set and X be the union of non-empty intervals $X_i = [\ell_i, r_i)$, $i \in I$, with $r_i \leq \ell_j$ for all $i < j$. Let $A = \{a_i \in \mathbb{R} \,|\, i \in I\}$ be such that $\beta X_i - a_i \subseteq X$ for all i. Then the transformation $T : X \to X$ defined by $Tx = \beta x - a_i$ for all $x \in X_i$, $i \in I$, is called a *right-continuous β-transformation*.

The corresponding *left-continuous β-transformation* $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is defined by $\widetilde{T}x = \beta x - a_i$ for all $x \in \widetilde{X}_i$, where $\widetilde{X}$ is the (disjoint) union of the intervals $\widetilde{X}_i = (\ell_i, r_i]$, $i \in I$. For $x \in \widetilde{X}$, the digit sequence generated by $\widetilde{T}$ is denoted by $\tilde{b}(x)$.

Observe that we have Lemma 5.2.1 also for $\widetilde{X}$ and the sequences $\tilde{b}(x)$. Then we have the following characterization of T-expansions and $\widetilde{T}$-expansions. Note that $b_1(\ell_i) = \tilde{b}_1(r_i) = i$ for all $i \in I$.

Theorem 5.2.1. *Let T be a right-continuous β-transformation and $\widetilde{T}$ the corresponding left-continuous β-transformation. Then a sequence $u = u_1 u_2 \cdots \in I^\omega$, is T-admissible if and only if*

$$b(\ell_{u_k}) \preceq u_k u_{k+1} \cdots \prec \tilde{b}(r_{u_k}) \text{ for all } k \geq 1. \tag{5.5}$$

A sequence $u = u_1 u_2 \cdots \in I^\omega$, is $\widetilde{T}$-admissible if and only if

$$b(\ell_{u_k}) \prec u_k u_{k+1} \cdots \preceq \tilde{b}(r_{u_k}) \text{ for all } k \geq 1.$$

Proof. We will prove only the first statement, since the proof of the second one is very similar.

If $u = b(x)$ for some $x \in X$, then $T^{k-1}x \in [\ell_{u_k}, r_{u_k})$ and $b(T^{k-1}x) = u_k u_{k+1} \cdots$ for all $k \geq 1$. By Lemma 5.2.1, we obtain immediately that $b(\ell_{u_k}) \preceq b(T^{k-1}x)$. We show that $b(T^{k-1}x) \prec \tilde{b}(r_{u_k})$. Since $T^{k-1}x < r_{u_k}$, there must be an index n, such that $b_n(T^{k-1}x) \neq \tilde{b}_n(r_{u_k})$ and $b_i(T^{k-1}x) = \tilde{b}_i(r_{u_k})$ for all $i < n$. Thus $T^{n+k-2}x < \widetilde{T}^{n-1} r_{u_k}$, which implies that $b_n(T^{k-1}x) < \tilde{b}_n(r_{u_k})$. Hence, $b(T^{k-1}x) \prec \tilde{b}(r_{u_k})$ and (5.5) holds.

For the other implication, suppose that u satisfies (5.5) and set for each $k \geq 1$, $x_k = \sum_{n=k}^\infty \frac{a_{u_n}}{\beta^{n-k+1}}$. We will show that $x_1 \in X$ and $u = b(x_1)$. First we show that $x_k < r_{u_k}$ for all $k \geq 1$. Since $u_k = \tilde{b}_1(r_{u_k})$, there exists some $s(k) > k$ such that $u_k \cdots u_{s(k)-1} = \tilde{b}_1(r_{u_k}) \cdots \tilde{b}_{s(k)-k}(r_{u_k})$ and $u_{s(k)} < \tilde{b}_{s(k)-k+1}(r_{u_k})$. Then we have

$$r_{u_k} - x_k = \frac{\widetilde{T}^{s(k)-k} r_{u_k}}{\beta^{s(k)-k}} - \sum_{n=s(k)}^{\infty} \frac{a_{u_n}}{\beta^{n-k+1}} > \frac{\ell_{\tilde{b}_{s(k)-k+1}(r_{u_k})} - x_{s(k)}}{\beta^{s(k)-k}} \geq \frac{r_{u_{s(k)}} - x_{s(k)}}{\beta^{s(k)-k}}$$

for all $k \geq 1$. By iterating $s^n(k) = s(s^{n-1}(k)) > s^{n-1}(k)$, $n \geq 1$, we obtain

$$r_{u_k} - x_k > \lim_{n\to\infty} \frac{r_{u_{s^n(k)}} - x_{s^n(k)}}{\beta^{s^n(k)-k}} = 0,$$

where we have used that the set $\{x_k \mid k \geq 1\}$ is bounded and that $s^n(k) \to \infty$, when $n \to \infty$. Similarly, we can show that $x_k \geq \ell_{u_k}$ for all $k \geq 1$. Therefore $x_1 \in [\ell_{u_1}, r_{u_1}) \subset X$, and we obtain inductively that $b_k(x_1) = u_k$ and $T^k x_1 = x_{k+1} \in [\ell_{u_{k+1}}, r_{u_{k+1}})$ for all $k \geq 1$. This proves the theorem. □

Example 5.2.1. Consider the classical greedy β-transformation, given by $T_\beta\, x = \beta x \pmod 1$. This fits in the above framework if we take $I = A = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ and set $X_i = \left[\frac{i}{\beta}, \frac{i+1}{\beta}\right)$ for $i \in \{0, 1, \ldots, \lfloor\beta\rfloor - 1\}$ and $X_{\lfloor\beta\rfloor} = \left[\frac{\lfloor\beta\rfloor}{\beta}, 1\right)$. The characterization that Parry gave of all the expansions generated by T_β, see Theorem 2.1.1, depends only on the $\widetilde{T}_\beta$-expansion of 1. We can see this from Theorem 5.2.1, since $T_\beta\, \ell_i = 0$ for all $i \in I$ and $\widetilde{T}_\beta\, r_i = 1$ for all $i \neq \lfloor\beta\rfloor$. In this setting the transformation $\widetilde{T}_\beta$ is sometimes called the *quasi-greedy β-transformation*.

To obtain the lazy transformation with digit set $A_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ we take $I = A = A_\beta$ again. We set $l_0 = 0$, $l_i = r_{i-1} = \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - i}{\beta}$ for $1 \leq i \leq \lfloor\beta\rfloor$ and $r_{\lfloor\beta\rfloor} = \frac{\lfloor\beta\rfloor}{\beta-1}$ and then the lazy transformation is the left-continuous transformation on the set of intervals given by these points. For the lazy transformation a characterization of the sequences depends only on the T_β-expansion of $\frac{\lfloor\beta\rfloor}{\beta-1} - 1$.

Example 5.2.2. The greedy β-transformation with allowable digit set $A = \{a_0, \dots, a_m\}$ as defined in Definition 2.2.3 is obtained in the following way. Set $I = \{0, \dots, m\}$ and let $X_i = \left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\right)$ for $0 \le i \le m-1$ and $X_m = \left[\frac{a_m}{\beta}, \frac{a_m}{\beta-1}\right)$. In [Ped05] Pedicini gave a characterization of all the sequences produced by the greedy algorithm. From the above we see that only the sequences $\tilde{b}(r_i)$ play a role here.

Example 5.2.3. In [FS08] Frougny and Steiner gave some examples of specific transformations generating minimal weight expansions. These transformations are symmetric (up to the endpoints of the intervals) and depend on two parameters, $\beta > 1$ and α, which lies in an interval depending on β. The transformations fit the above framework by taking $I = \{-1, 0, 1\}$, $a_i = i$ and setting $X_{-1} = [-\beta\alpha, -\alpha)$, $X_0 = [-\alpha, \alpha)$ and $X_1 = [\alpha, \beta\alpha)$. Suppose that an $x \in X$ has a finite expansion, i.e., suppose that there is an $N \ge 1$ such that $b_n(x) = 0$ for all $n > N$. Then the absolute sum of digits of x is $\sum_{k=1}^{N} |b_k(x)|$. The transformations from [FS08] generate expansions of minimal weight in the following sense. If $x \in \mathbb{Z}[1/\beta]$, then the absolute sum of digits of the expansion of x that such a transformation generates, is less than or equal to that of all possible other expansions of x in base β with integer digits.

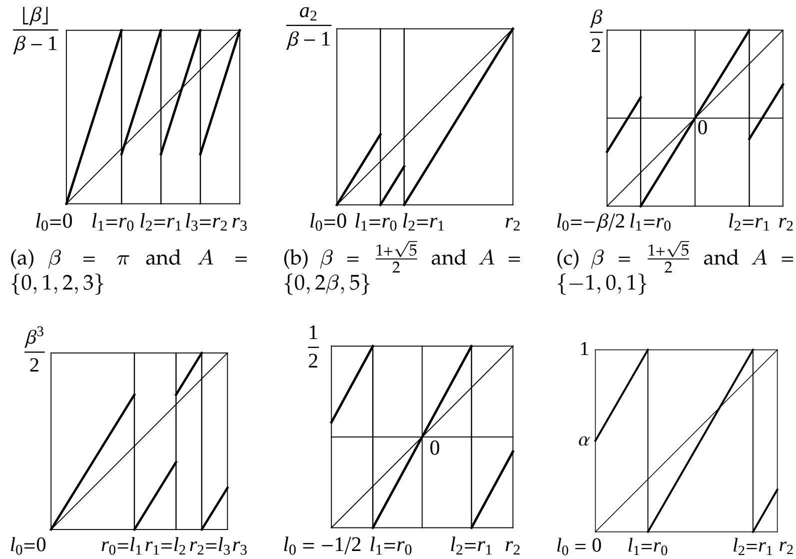
Example 5.2.4. In Section 3.3.3 we saw the transformation U, defined from the interval $[0, \beta^3/2)$ to itself. If we take β equal to the golden mean, $I = \{0, 1, 2, 3\}$, $A = \{0, \beta, \beta/2, 3\beta/2 + 1/2\}$ and set $X_0 = [0, 1)$, $X_1 = [1, 3/2)$, $X_2 = [3/2, 1 + \beta/2)$ and $X_3 = [1 + \beta/2, \beta^3/2)$, then we see that U is also a transformation from the class defined above.

Example 5.2.5. In [AS07] Akiyama and Scheicher defined the *symmetric β-transformations*. For $1 < \beta < 3$ they can be obtained by taking $I = A = \{-1, 0, 1\}$ and setting $X_{-1} = \left[-\frac{1}{2}, -\frac{1}{2\beta}\right)$, $X_0 = \left[-\frac{1}{2\beta}, \frac{1}{2\beta}\right)$ and $X_1 = \left[\frac{1}{2\beta}, \frac{1}{2}\right)$. Then $Tx = \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$.

Example 5.2.6. The *linear mod 1 transformations* are maps from the interval $[0, 1)$ to itself, given by $Tx = \beta x + \alpha \pmod 1$ with $\beta > 1$ and $0 \le \alpha < 1$. They are well-studied, see for example [Hof81] by Hofbauer and [FL97b], [FL97a] and [FL96] by Flatto and Lagarias. Note that if we set $\alpha = 0$, then this is the classical greedy β-transformation on the interval $[0, 1)$.

We can obtain these transformations in the following way. Let $n \ge 1$ be such that $n < \beta + \alpha \le n + 1$. Then take $I = \{0, \dots, n\}$ and set $A = \{a_i = i - \alpha \,|\, i \in I\}$. Define $X_0 = \left[0, \frac{1-\alpha}{\beta}\right)$, $X_n = \left[\frac{n-\alpha}{\beta}, 1\right)$ and for $1 \le i \le n-1$, $X_i = \left[\frac{i-\alpha}{\beta}, \frac{i+1-\alpha}{\beta}\right)$. Then $Tx = \beta x + \alpha - i$ on X_i, $i \in I$.

In Figure 5.1 we see examples of these transformations. What all these transformations have in common, but what is not assumed in general, is that their domain X is an interval.



(a) $\beta = \pi$ and $A = \{0, 1, 2, 3\}$

(b) $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{0, 2\beta, 5\}$

(c) $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{-1, 0, 1\}$

(d) $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{0, \beta, \frac{\beta}{2}, \frac{3\beta+1}{2}\}$

(e) $\beta = t$ and $A = \{-1, 0, 1\}$

(f) $\beta = \sqrt{3}$ and $A = \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$

Figure 5.1 In (a) we see a classical lazy transformation, (b) shows a greedy transformation with arbitrary digits, (c) is a minimal weight transformation, (d) is the transformation from Figure 3.7, in (e) we see a symmetric β-transformation with β equal to the Tribonacci number, t, and in (f) there is a linear mod 1 transformation.

We will use the set of T-admissible sequences to construct a natural extension and a multiple tiling for T. Therefore, we need to define the following sets. Similarly to the β-shift, we define the *T-shift* $\bar{\mathcal{S}}$ as the set of sequences $u \in I^{\mathbb{Z}}$ such that every finite subsequence of u occurs in some T-admissible sequence. Then we have

$$\begin{aligned}\bar{\mathcal{S}} &= \{(u_k)_{k\in\mathbb{Z}} \in I^{\mathbb{Z}} \mid b(\ell_{u_k}) \preceq u_k u_{k+1} \cdots \preceq \tilde{b}(r_{u_k}) \text{ for all } k \in \mathbb{Z}\} \\ &= \{(u_k)_{k\in\mathbb{Z}} \mid u_k u_{k+1} \cdots \in b(X) \cup \tilde{b}(\widetilde{X}) \text{ for all } k \in \mathbb{Z}\}.\end{aligned}$$

The compact shift $\bar{\mathcal{S}}$ is the closure of the non-compact shift

$$\begin{aligned}\mathcal{S} &= \{(u_k)_{k\in\mathbb{Z}} \in I^{\mathbb{Z}} \mid b(\ell_{u_k}) \preceq u_k u_{k+1} \cdots \prec \tilde{b}(r_{u_k}) \text{ for all } k \in \mathbb{Z}\} \qquad (5.6) \\ &= \{(u_k)_{k\in\mathbb{Z}} \mid u_k u_{k+1} \cdots \in b(X) \text{ for all } k \in \mathbb{Z}\}.\end{aligned}$$

5.3 Another natural extension

5.3.1 *Do not forget the past*

Our goal in this section is to define a measure theoretical natural extension for the class of transformations defined in the previous section, under suitable assumptions on β and the digit set. This natural extension will allow us to define a multiple tiling of some Euclidean space. The set-up for this multiple tiling is the same as the one Thurston gave in [Thu89] for the classical greedy β-transformation $T_\beta x = \beta x \,(\mathrm{mod}\, 1)$. We will give this set-up here.

From now on, let $\beta > 1$ be a Pisot unit with minimal polynomial $x^d - c_1 x^{d-1} - \cdots - c_d$ and $\beta_2, \ldots, \beta_d$ its Galois conjugates. Thus, $|\beta_j| < 1$, $c_k \in \mathbb{Z}$ for $1 \le k \le d$ and $|c_d| = 1$. Set $\beta_1 = \beta$. Let M_β be the companion matrix

$$M_\beta = \begin{pmatrix} c_1 & c_2 & \cdots & c_{d-1} & c_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and let $\mathbf{v}_j = v_j(\beta_j^{d-1}, \ldots, \beta_j, 1)^t$ with $v_j \in \mathbb{C}$, $1 \le j \le d$, be the right eigenvectors of M_β such that $\mathbf{v}_1 + \cdots + \mathbf{v}_d = (1, 0, \ldots, 0)^t$. If $\beta_j \in \mathbb{R}$, then $\mathbf{v}_j \in \mathbb{R}^d$. If $\beta_j \in \mathbb{C}$ and β_k is its complex conjugate, then the entries of $\mathbf{v}_k$ are the complex conjugates of the entries of $\mathbf{v}_j$. Since $|\det M_\beta| = |c_d| = 1$, M_β^{-1} is a matrix with integer entries. Hence, we have for all $k \in \mathbb{Z}$,

$$\beta^k \mathbf{v}_1 + \beta_2^k \mathbf{v}_2 + \cdots + \beta_d^k \mathbf{v}_d = M_\beta^k(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_d) = M_\beta^k (1, 0, \ldots, 0)^t \in \mathbb{Z}^d. \quad (5.7)$$

Let H be the hyperplane of $\mathbb{R}^d$ which is spanned by the real and imaginary parts of $\mathbf{v}_2, \ldots, \mathbf{v}_d$. Then $H \simeq \mathbb{R}^{d-1}$.

Let $A \subset \mathbb{Q}(\beta)$. Since I is finite, there is a $q \in \mathbb{Z}$, such that $A \subseteq q^{-1}\mathbb{Z}[\beta]$. Let $\Gamma_j : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_j)$, $1 \le j \le d$, be the isomorphism defined by $\Gamma_j(\beta) = \beta_j$ and $\Gamma_j(q) = q$ for $q \in \mathbb{Q}$. Then we define the following functions. For sequences $w = (w_k)_{k \le 0} \in {}^\omega I$, set

$$\varphi(w) = \sum_{k \le 0} \sum_{j=2}^{d} \Gamma_j(a_{w_k}) \beta_j^{-k} \mathbf{v}_j \in H. \quad (5.8)$$

For finite sequences $v_1 \cdots v_n \in I^n$, $n \ge 1$, define

$$\varphi(v_1 \cdots v_n) = \sum_{k=1}^{n} \sum_{j=2}^{d} \Gamma_j(a_{v_k}) \beta_j^{n-k} \mathbf{v}_j.$$

Thus $\varphi(w) = \lim_{n \to \infty} \varphi(w_{-n+1} \cdots w_0)$. For two-sided infinite sequences $u = (u_k)_{k \in \mathbb{Z}} \in I^{\mathbb{Z}}$, we set

$$\begin{aligned} \psi(u) &= (.u_1 u_2 \cdots)\mathbf{v}_1 - \varphi(\cdots u_{-1} u_0) = \sum_{k=1}^{\infty} \frac{a_{u_k}}{\beta^k} \mathbf{v}_1 - \varphi(\cdots u_{-1} u_0) \\ &= \sum_{k \ge 1} a_{u_k} \beta^{-k} \mathbf{v}_1 - \sum_{k \le 0} \sum_{j=2}^{d} \Gamma_j(a_{u_k}) \beta_j^{-k} \mathbf{v}_j \in \mathbb{R}^d. \end{aligned}$$

Define the set $\widehat{X} = \psi(\mathcal{S})$. This will be (up to sets of measure zero) our natural extension domain. We can write $\widehat{X}$ as the disjoint union of the sets $\widehat{X}_i = \psi(\{(u_k)_{k \in \mathbb{Z}} \in \mathcal{S} \,|\, u_1 = i\})$, $i \in I$. The natural extension transformation

$\widehat{T} : \widehat{X} \to \widehat{X}$ is defined piecewise on the sets $\widehat{X}_i$ by

$$\widehat{T}\mathbf{x} = M_\beta \mathbf{x} - \sum_{j=1}^{d} \Gamma_j(a_i)\mathbf{v}_j, \quad \text{if } \mathbf{x} \in \widehat{X}_i.$$

We will show that $\widehat{T}$ is λ^d a.e. invertible on $\widehat{X}$, where λ^d is the d-dimensional Lebesgue measure. The next lemma says that $\widehat{T} : \widehat{X} \to \widehat{X}$ is surjective.

Lemma 5.3.1. *For all $\mathbf{x} \in \widehat{X}$ there is an $\mathbf{x}' \in \widehat{X}$, such that $\widehat{T}\mathbf{x}' = \mathbf{x}$. Hence $\widehat{T}\widehat{X} = \widehat{X}$.*

Proof. Let $\mathbf{x} \in \widehat{X}$. Then there is a sequence $u = (u_n)_{n\in\mathbb{Z}} \in \mathcal{S}$ with $\mathbf{x} = \psi(u)$. Let σ^{-1} denote the right-shift on $I^{\mathbb{Z}}$, i.e., $\sigma^{-1}(u) = w$ with $w_k = u_{k-1}$ for all $k \in \mathbb{Z}$. Then $\sigma^{-1}(u) \in \mathcal{S}$. Set $\mathbf{x}' = \psi(\sigma^{-1}(u))$. Then

$$\begin{aligned}
\widehat{T}\mathbf{x}' &= M_\beta\Big(\sum_{k=1}^{\infty} \frac{a_{u_{k-1}}}{\beta^k}\mathbf{v}_1 - \varphi(\cdots u_{-2}u_{-1})\Big) - \sum_{j=1}^{d}\Gamma_j(a_{u_0})\mathbf{v}_j \\
&= \sum_{k=1}^{\infty} \frac{a_{u_k}}{\beta^k}\mathbf{v}_1 + a_{u_0}\mathbf{v}_1 - \sum_{k\le -1}\sum_{j=2}^{d}\Gamma_j(a_{u_{-k}})\beta_j^{-k}\mathbf{v}_j - \sum_{j=1}^{d}\Gamma_j(a_{u_0})\mathbf{v}_j \\
&= (.u_1u_2\cdots)\mathbf{v}_1 - \varphi(\cdots u_{-1}u_0) = \psi(u) = \mathbf{x}.
\end{aligned}$$

Hence, $\widehat{T} : \widehat{X} \to \widehat{X}$ is surjective. $\square$

Since $\widehat{X} \subset \mathbb{R}\mathbf{v}_1 + H$, we can write each element $\mathbf{x} \in \widehat{X}$ as $\mathbf{x} = x\mathbf{v}_1 - \mathbf{y} = x\mathbf{v}_1 - \sum_{j=2}^{d} y_j\mathbf{v}_j$ with $x \in X$, $\mathbf{y} \in H$ and $y_j \in \mathbb{C}$, $2 \le j \le d$. Then we have

$$\widehat{T}\Bigg(x\mathbf{v}_1 - \sum_{j=2}^{d} y_j\mathbf{v}_j\Bigg) = (Tx)\mathbf{v}_1 - \sum_{j=2}^{d}\Big(\beta_j y_j + \Gamma_j(a_{b_1(x)})\Big)\mathbf{v}_j.$$

Define the projection $\pi : \widehat{X} \to X$ by $\pi(x\mathbf{v}_1 - \mathbf{y}) = x$. Then, $T \circ \pi = \pi \circ \widehat{T}$.

Next we show that $\widehat{X}$ is a Lebesgue measurable set. Since $\widehat{X}$ is not closed, we consider $\widehat{Y} = \psi(\bar{\mathcal{S}})$ first.

Lemma 5.3.2. *$\widehat{Y}$ is compact and thus Lebesgue measurable.*

Proof. Note that $\bar{\mathcal{S}}$ is a closed subset of a compact metric space ${}^{\omega}I$. Hence, $\bar{\mathcal{S}}$ is compact. Since $\psi : \bar{\mathcal{S}} \to \mathbb{R}^d$ is a continuous map, we also have that $\widehat{Y} = \psi(\bar{\mathcal{S}})$ is compact. $\square$

A sequence $(u_k)_{k\in\mathbb{Z}} \in \bar{\mathcal{S}}$ is not in $\mathcal{S}$ if and only if $u_ku_{k+1}\cdots = \tilde{b}(r_i)$ for some $i \in I$, $k \in \mathbb{Z}$. Therefore, the projection of $\widehat{Y}\backslash\widehat{X}$ on the line $\mathbb{R}\mathbf{v}_1$ along H is a countable set and $\lambda^d(\widehat{Y}\backslash\widehat{X}) = 0$. This implies that $\widehat{X}$ is a Lebesgue measurable set with $\lambda^d(\widehat{X}) = \lambda^d(\widehat{Y})$.

Let $\mathcal{L}$ be the Lebesgue σ-algebra on X and $\widehat{\mathcal{L}}$ the Lebesgue σ-algebra on $\widehat{X}$. We want to prove that the system $(\widehat{X}, \widehat{\mathcal{L}}, \lambda^d, \widehat{T})$ is, up to sets of measure zero, a version of the natural extension of the system $(X, \mathcal{L}, \mu, T)$, where the measure μ on $(X, \mathcal{L})$ is defined by $\mu = \lambda^d \circ \pi^{-1}$. For the precise definition of a natural extension, see Definition 3.1.1. We first need the following lemma to establish the invertibility of $\widehat{T}$ off of a set of measure zero.

Lemma 5.3.3. *For all $i, i' \in I$ with $i \neq i'$, we have $\lambda^d(\widehat{T}\widehat{X}_i \cap \widehat{T}\widehat{X}_{i}') = 0$.*

Proof. Recall that $|\det M_\beta| = 1$. Then, by the definition of $\widehat{T}$, we have for all $i \in I$ that $\lambda^d(\widehat{T}\widehat{X}_i) = |\det M_\beta|\lambda^d(\widehat{X}_i) = \lambda^d(\widehat{X}_i)$. Since $\widehat{T}\widehat{X} = \widehat{X}$, then

$$\sum_{i\in I}\lambda^d(\widehat{T}\widehat{X}_i) = \sum_{i\in I}\lambda^d(\widehat{X}_i) = \lambda^d(\widehat{X}) = \lambda^d(\widehat{T}\widehat{X}) = \lambda^d\Big(\bigcup_{i\in I}\widehat{T}\widehat{X}_i\Big),$$

which proves the lemma. □

Let

$$\widehat{N} = \bigcup_{n\in\mathbb{Z}}\widehat{T}^n\left(\bigcup_{i,i'\in I, i\neq i'}\widehat{T}\widehat{X}_i \cap \widehat{T}\widehat{X}_{i'}\right).$$

Then by Lemma 5.3.3, $\lambda^d(\widehat{N}) = 0$. Set $\widehat{E} = \widehat{X}\backslash\widehat{N}$. Then $\lambda^d(\widehat{E}) = \lambda^d(\widehat{X})$ and $\widehat{T}$ is a bijection on $\widehat{E}$. Let $\mathcal{E}$ be $\widehat{\mathcal{L}} \cap \widehat{E}$ and let $\lambda_{\widehat{E}}$ be the restriction of λ^d to $\widehat{E}$. Then $\widehat{T}$ is an invertible, measure preserving transformation on $(\widehat{E}, \mathcal{E}, \lambda_{\widehat{E}})$.

The measure μ, defined by $\mu = \lambda^d \circ \pi^{-1}$ is an invariant measure for T and its support is contained in X. The projection map π is measurable and measure preserving and we have that $(T \circ \pi)(\mathbf{x}) = (\pi \circ \widehat{T})(\mathbf{x})$ for all $\mathbf{x} \in \widehat{E}$. Let $M = \{x \in X \mid \pi^{-1}\{x\} \subseteq \widehat{N}\}$. Then $T(X\backslash M) \subseteq X\backslash M$ and $\mu(M) = (\lambda^d \circ \pi^{-1})(M) \leq \lambda^d(\widehat{N}) = 0$. Since π is surjective from $\widehat{E}$ to $X\backslash M$, π is a factor map from $(\widehat{E}, \mathcal{E}, \lambda_{\widehat{E}}, \widehat{T})$ to $(X, \mathcal{B}, \mu, T)$. This gives us (ne1)-(ne4) from Definition 3.1.1. In the next theorem we prove (ne5).

Theorem 5.3.1. *Let T be a right-continuous β-transformation as in Definition 5.2.3 with β a Pisot unit and $A \subset \mathbb{Q}(\beta)$. Then the dynamical system $(\widehat{E}, \mathcal{E}, \lambda_{\widehat{E}}, \widehat{T})$ is a natural extension of the dynamical system $(X, \mathcal{L}, \mu, T)$.*

Proof. We have already shown that $\widehat{T}$ is invertible on $\widehat{E}$ and that π is a factor map from $(\widehat{E}, \mathcal{E}, \lambda_{\widehat{E}}, \widehat{T})$ to $(X, \mathcal{L}, \mu, T)$. The only thing that remains in order to get the theorem is that

$$\bigvee_{k\geq 0}\widehat{T}^k\pi^{-1}(\mathcal{L}) = \mathcal{E}.$$

By the definition of $\mathcal{S}$, it is clear that $\bigvee_{k\geq 0}\widehat{T}^k\pi^{-1}(\mathcal{L}) \subseteq \mathcal{E}$. To show the other inclusion, we use Theorem 3.1.1. Take $x\mathbf{v}_1 - \mathbf{y}$, $x'\mathbf{v}_1 - \mathbf{y}' \in \widehat{E}$. Suppose that $x \neq x'$. Then there are two disjoint intervals $B, B' \subseteq X\backslash M$ with $x \in B$ and $x' \in B'$. But then also $x\mathbf{v}_1 - \mathbf{y} \in \pi^{-1}(B)$ and $x'\mathbf{v}_1 - \mathbf{y}' \in \pi^{-1}(B')$ and these two sets are disjoint. Now, let $x\mathbf{v}_1 - \mathbf{y}, x'\mathbf{v}_1 - \mathbf{y}' \in \widehat{E}$ with $x = x'$ and $\mathbf{y} \neq \mathbf{y}'$. The fact that $\mathbf{y} \neq \mathbf{y}'$ implies that there are sequences $w \cdot b(x), w' \cdot b(x) \in \mathcal{S}$, such that $\mathbf{y} = \varphi(w) \neq \varphi(w') = \mathbf{y}'$. Let $n \geq 1$ be the first index such that $w_{-n+1} \neq w'_{-n+1}$. Let

$$x_n = \sum_{k=1}^{n}\frac{a_{w_{-n+k}}}{\beta^k} + \frac{x}{\beta^n} \quad \text{and} \quad x'_n = \sum_{k=1}^{n}\frac{a_{w'_{-n+k}}}{\beta^k} + \frac{x}{\beta^n}.$$

Then $x_n \neq x'_n$, so there exist two disjoint intervals B, $B' \subseteq X\backslash M$, such that $x_n \in B$ and $x'_n \in B'$. Moreover, $x\mathbf{v}_1 - \mathbf{y} \in \widehat{T}^n\pi^{-1}(B)$ and $x\mathbf{v}_1 - \mathbf{y}' \in \widehat{T}^n\pi^{-1}(B')$.

By the invertibility of $\widehat{T}$, the sets $\widehat{T}^n\pi^{-1}(B)$ and $\widehat{T}^n\pi^{-1}(B')$ are disjoint, hence $x\mathbf{v}_1 - \mathbf{y}$ and $x\mathbf{v}_1 - \mathbf{y}'$ are in two disjoint elements of $\widehat{T}^n\pi^{-1}(\mathcal{L})$. This shows that $\bigvee_{k\geq 0}\widehat{T}^k\pi^{-1}(\mathcal{L}) = \mathcal{E}$ and thus that $(\widehat{E}, \mathcal{E}, \lambda_{\widehat{E}}, \widehat{T})$ is a natural extension of $(X, \mathcal{L}, \mu, T)$. □

5.3.2 *In shape*

To obtain multiple tilings from the natural extensions defined in the previous section, we need to have a closer look at the shape of the natural extension domain. In this section we assume that $T : X \to X$ is a right-continuous β-transformation as in Definition 5.2.3 with $A \subset \mathbb{Z}[\beta]$ and that T is surjective, i.e., that $TX = X$. We first want to prove that $\widehat{Y}$ covers the torus, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For that, we need the following lemma.

Lemma 5.3.4. *The set $\bigcup_{n\geq 0}\beta^n X\mathbf{v}_1$ is dense in $\mathbb{T}^d$, i.e. for all $\mathbf{u} \in [0,1)^d$ and all $\varepsilon > 0$ there is an $n \geq 0$ and an $x \in \beta^n X$ with $\beta^n x\mathbf{v}_1 \equiv \mathbf{x} \pmod{\mathbb{Z}^d}$ and $\|\mathbf{u} - \mathbf{x}\| < \varepsilon$.*

Proof. Since the coefficients of $\mathbf{v}_1$ are linearly independent over $\mathbb{Q}$, the set $\mathbb{R}\mathbf{v}_1$ is dense in $\mathbb{T}^d$ by the Kronecker Approximation Theorem. Hence, for each $\varepsilon > 0$ and each $\mathbf{u} \in [0,1)^d$, there is an $x \in \mathbb{R}$ with $x\mathbf{v}_1 \equiv \mathbf{x} \,(\mathrm{mod}\, \mathbb{Z}^d)$ and $\|\mathbf{u} - \mathbf{x}\| < \varepsilon$.

Fix $\varepsilon > 0$ and consider the finite set of points

$$B = \{(n_1\varepsilon, \ldots, n_d\varepsilon) \in [0,1)^d \mid n_j \in \mathbb{N},\ 1 \leq j \leq d\}.$$

For each N, let η_N denote the maximal distance between B and the set $[0,N)\mathbf{v}_1 \,(\mathrm{mod}\, Z^d)$. Then $\eta_N \downarrow 0$, as $N \to \infty$. So, there is an $N = N(\varepsilon)$, such that $\eta_N < \varepsilon$. Since each $\mathbf{u} \in [0,1)^d$ is within distance $c \cdot \varepsilon$ of B for a certain constant c, we have that for each $\mathbf{u} \in [0,1)^d$ there is an $x \in [0, N(\varepsilon))$ with $x\mathbf{v}_1 \equiv \mathbf{x} \,(\mathrm{mod}\, \mathbb{Z}^d)$ and $\|\mathbf{u} - \mathbf{x}\| < (1 + c)\varepsilon$.

Let $t \in \mathbb{R}$ and consider the lattice $t\mathbf{v}_1 + \mathbb{Z}^d$ in $\mathbb{R}^d$. Then every point $\mathbf{y} \in \mathbb{R}^d$ can be written as $\mathbf{y} = t\mathbf{v}_1 + \mathbf{z} + \mathbf{y}'$ with $\mathbf{z} \in \mathbb{Z}^d$ and $\mathbf{y}' \in [0,1)^d$. By the same reasoning as above, for every point $\mathbf{u} \in t\mathbf{v}_1 + [0,1)^d$ there is an $x \in [t, t + N(\epsilon))$ and an $\mathbf{x} \in \mathbb{R}^d$ with $x\mathbf{v}_1 - \mathbf{x} \equiv t\mathbf{v}_1 \pmod{\mathbb{Z}^d}$ and $\|\mathbf{u} - \mathbf{x}\| < (1 + c)\varepsilon$.

Since $\bigcup_{n\geq 0}\beta^n X$ contains arbitrarily large intervals, there is an $n \geq 0$ and a $t \in \mathbb{R}$, such that $[t, t + N(\varepsilon)) \subset \beta^n X$. This proves the lemma. □

Lemma 5.3.5. *If $A \subset \mathbb{Z}[\beta]$, then $\widehat{Y}$ covers $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, i.e., $\widehat{Y} + \mathbb{Z}^d = \mathbb{R}^d$.*

Proof. We show that for each $\mathbf{u} \in \mathbb{T}^d$ and each $\varepsilon > 0$, there is an element $\psi(u) \in \widehat{Y}$, such that the distance from $\mathbf{u}$ to $\psi(u)$ in $\mathbb{T}^d$ is less than 2ε. Therefore, let $\mathbf{u} \in \mathbb{T}^d$ and let $\varepsilon > 0$ be given. Let $k \geq 0$ be such that $\beta^k X$ contains an interval of length $N(\varepsilon)$, with $N(\varepsilon)$ as given in Lemma 5.3.4, and such that $\|M_\beta^k\varphi(w)\| < \varepsilon$ for all $w \in {}^\omega I$. By Lemma 5.3.4, there is an $x \in \beta^k X$ with $x\mathbf{v}_1 \equiv \mathbf{x} \pmod{\mathbb{Z}^d}$ and $\|\mathbf{u} - \mathbf{x}\| < \varepsilon$.

Since $\frac{x}{\beta^k} \in X$ and $TX = X$, there is a sequence $u \in \mathcal{S}$ with $u_{-k+1}u_{-k+2}\cdots = b(x/\beta^k)$. Take such a sequence u and consider $\psi(u)$. By (5.7) we have for all

$n \in \mathbb{Z}$ that

$$\beta^n \mathbf{v}_1 = -\sum_{j=2}^{d} \beta_j^n \mathbf{v}_j + M_\beta^n (1, 0, \ldots, 0)^t \equiv -\sum_{j=2}^{d} \beta_j^n \mathbf{v}_j \pmod{\mathbb{Z}^d}$$

and if we write $a_i = q_{d-1}\beta^{d-1} + \cdots + q_1\beta + q_0$ for some $q_\ell \in \mathbb{Z}$, then

$$\begin{aligned} a_i \beta^n \mathbf{v}_1 &= \sum_{\ell=0}^{d-1} q_\ell \beta^{\ell+n} \mathbf{v}_1 = -\sum_{j=2}^{d} \sum_{\ell=0}^{d-1} q_\ell \beta_j^{\ell+n} \mathbf{v}_j + M_\beta^{\ell+n} (1, 0, \ldots, 0)^t \\ &\equiv -\sum_{j=2}^{d} \Gamma_j(a_i) \beta_j^n \mathbf{v}_j \pmod{\mathbb{Z}^d}. \end{aligned}$$

Hence, we also have $\psi(u) \equiv \sum_{n\in\mathbb{Z}} a_{u_n} \beta^{-n} \mathbf{v}_1 \pmod{\mathbb{Z}^d}$ and for each $k \geq 0$,

$$\psi(u) \equiv \sum_{n>-k} a_{u_n} \beta^{-n} \mathbf{v}_1 - M_\beta^k \varphi(\cdots u_{-k-1} u_{-k}) \pmod{\mathbb{Z}^d}.$$

Thus, $\psi(u) \equiv x\mathbf{v}_1 - M_\beta^k \varphi(\cdots u_{-k-1} u_{-k}) \pmod{\mathbb{Z}^d}$. This implies that there is a $\mathbf{y} \in \mathbb{R}^d$ with $\psi(u) \equiv \mathbf{y} \pmod{\mathbb{Z}^d}$ and $\|\mathbf{u} - \mathbf{y}\| \leq \|\mathbf{u} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| < 2\varepsilon$. Since $\widehat{Y}$ is compact, the lemma is proved. □

If $A \subset q^{-1}\mathbb{Z}[\beta]$, $q \in \mathbb{Z}$, then we obtain similarly that $\widehat{Y}$ covers $\mathbb{R}^d/(q^{-1}\mathbb{Z}^d)$. Then we can write $\mathbb{R}^d = \bigcup_{\mathbf{z}\in q^{-1}\mathbb{Z}^d} (\mathbf{z} + \widehat{Y})$. By Baire's Theorem it then follows that $\widehat{Y}$ is not nowhere dense, and since $\widehat{Y}$ is closed, this implies that $\lambda^d(\widehat{Y}) > 0$ and thus also $\lambda^d(\widehat{X}) > 0$.

We can write

$$\widehat{X} = \bigcup_{x\in X} (x\mathbf{v}_1 - \mathcal{D}_x) \quad \text{with} \quad \mathcal{D}_x = \big\{\varphi(w) \,|\, w \cdot b(x) \in \mathcal{S}\big\}, \tag{5.9}$$

where φ is as in (5.8) and $\mathcal{S}$ as in (5.6). For the multiple tiling we will construct in Section 5.4, the prototiles will be the sets $\mathcal{D}_x$ for $x \in \mathbb{Z}[\beta] \cap X$. The next lemma says that these sets are compact.

Lemma 5.3.6. *Every set $\mathcal{D}_x$, $x \in X$, is compact.*

Proof. Let $x \in X$ and consider the subset $\mathcal{W} = \{w \in {}^{\omega}I \,|\, w \cdot b(x) \in \mathcal{S}\}$ of the compact space ${}^{\omega}I$. We want to show that $\mathcal{W}$ is closed and hence, compact. Therefore, take some converging sequence $(w^{(n)})_{n\geq 0} \subseteq \mathcal{W}$ and let $\lim_{n\to\infty} w^{(n)} = w$. This implies that we can find a subsequence $(w^{(n_k)})_{k\geq 0}$, such that $w_{-k}^{(n_k)} \cdots w_0^{(n_k)} = w_{-k} \cdots w_0$ for all $k \geq 0$. Since $w^{(n_k)} \cdot b(x) \in \mathcal{S}$ for all k, also $w \cdot b(x) \in \mathcal{S}$. Hence, $\mathcal{W}$ is closed. Since $\mathcal{D}_x$ is the image of the compact set $\mathcal{W}$ under the continuous map φ, it is compact as well. □

To distinguish different sets $\mathcal{D}_x$, we introduce the set $\mathcal{V}$ below. For every $i \in I$ with $r_i \in X$, let m_i be the minimal positive integer such that

$$\widetilde{T}^{m_i} r_i = T^{m_i} r_i,$$

with $m_i = \infty$ if $\widetilde{T}^k r_i \neq T^k r_i$ for all $k \geq 1$. Set

$$\mathcal{V} = \bigcup_{i\in I}\{\ell_i,\, r_i\} \cup \bigcup_{1\le k<m_i,\, i\in I:\, r_i\in X} \big\{\widetilde{T}^k r_i,\, T^k r_i\big\}. \tag{5.10}$$

Then $\mathcal{V}$ is a finite set if and only if $\tilde{b}(r_i)$ and $b(r_i)$ are eventually periodic or $m_i < \infty$ for every $r_i \in X$, $i \in I$. The restriction to $r_i \in X$ is justified by the following lemma, where we use the assumption $TX = X$. Note that $\{r_i \in X \,|\, i \in I\} = \{\ell_i \in \widetilde{X} \,|\, i \in I\}$.

Lemma 5.3.7. *For every $r_i \notin X$, $i \in I$, there exists a positive integer h and an $r_{i'} \in X$, $i' \in I$, such that $\widetilde{T}^h(r_{i'}) = r_i$, or $\tilde{b}(r_i)$ is purely periodic. For every $\ell_i \notin \widetilde{X}$, $i \in I$, there exists a positive integer h and an $r_{i'} \in X$, $i' \in I$, such that $T^h(r_{i'}) = \ell_i$ or $b(\ell_i)$ is purely periodic.*

Proof. Let $r_i \notin X$, $i \in I$. Since $\widetilde{T}\widetilde{X} = \widetilde{X}$, we have $r_i = \widetilde{T} r_{i'}$ for some $i' \in I$. If $r_{i'} \in X$, then the first statement is shown. Otherwise, we proceed recursively. We obtain that either $\widetilde{T}^{-h} r_i$ contains some $r_{i'} \in X$ for some $h \geq 1$, or that every set $\widetilde{T}^{-h} r_i$, $h \geq 0$, consists only of numbers $r_{i'} \notin X$. In the latter case, the finiteness of I implies that $\tilde{b}(r_{i'})$ is purely periodic for all these i'.

Since $\{r_i \in X \,|\, i \in I\} = \{\ell_i \in \widetilde{X} \,|\, i \in I\}$, the second statement is proved similarly. $\square$

Since the sets $\mathcal{D}_x$ for $x \in \mathbb{Z}[\beta] \cap X$ will be used as prototiles, we would like a condition under which there are only finitely many different sets $\mathcal{D}_x$. The next proposition is an important step in this direction.

Proposition 5.3.1. *If $x, y \in X$, $x < y$, and $(x, y\,] \cap \mathcal{V} = \emptyset$, then $\mathcal{D}_x = \mathcal{D}_y$.*

The main ingredient of the proof of Proposition 5.3.1 is the following simple lemma.

Lemma 5.3.8. *If $x \in X \cap \widetilde{X}$ and $\widetilde{T}^n x = T^n x$ for some $n \geq 1$, then*

$$\varphi\big(b_1(x)\cdots b_n(x)\big) = \varphi\big(\tilde{b}_1(x)\cdots \tilde{b}_n(x)\big).$$

Proof. We have

$$\sum_{k=1}^{n} a_{\tilde{b}_k(x)}\beta^{n-k} = \beta^n x - \widetilde{T}^n x = \beta^n x - T^n x = \sum_{k=1}^{n} a_{b_k(x)}\beta^{n-k}.$$

By applying Γ_j, $2 \leq j \leq d$, to this equation, the lemma is proved. $\square$

Lemma 5.3.9. *Let $x, y \in X$, $x < y$, $(x, y\,] \cap \mathcal{V} = \emptyset$, and suppose that there is a finite sequence $w_{-n}\cdots w_0$, such that $w_{-n}\cdots w_0 b(x)$ is T-admissible, but the sequence $w_{-n}\cdots w_0 b(y)$ is not. Then there exists some sequence $w'_{-n}\cdots w'_0$ such that $w'_{-n}\cdots w'_0 b(y)$ is admissible and*

$$\varphi(w'_{-n}\cdots w'_0) = \varphi(w_{-n}\cdots w_0) + \mathcal{O}(\rho^n), \tag{5.11}$$

where $\rho = \max_{2\le j\le d}|\beta_j| < 1$ and the constant implied by the $\mathcal{O}$-symbol depends only on T.

Proof. Let k, $0 \le k \le n$, be maximal such that

$$w_{-k} \cdots w_0 = \tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}).$$

The admissibility of $w_{-n} \cdots w_0 b(x)$ implies that for each $0 \le i \le n$,

$$b(\ell_{w_{-i}}) \preceq w_{-i} \cdots w_0 b(x) \prec w_{-i} \cdots w_0 b(y).$$

For $k < i \le n$, we have by the maximality of k, that

$$w_{-i} \cdots w_0 \ne \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}}),$$

hence $w_{-i} \cdots w_0 \prec \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$. This implies that $w_{-i} \cdots w_0 b(y) \prec \tilde{b}(r_{w_{-i}})$. The non-admissibility of $w_{-n} \cdots w_0 b(y)$ means therefore that for some i, $0 \le i \le k$, we have $w_{-i} \cdots w_0 b(y) \succeq \tilde{b}(r_{w_{-i}})$. Since $w_{-i} \cdots w_0 b(x) \prec \tilde{b}(r_{w_{-i}})$, then $w_{-i} \cdots w_0 = \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$. Thus

$$x < \widetilde{T}^{k+1} r_{w_{-k}} \le \widetilde{T}^{i+1} r_{w_{-i}} \le y.$$

In order to prove the lemma, we want to replace the sequence $w_{-k} \cdots w_0$ by a suitable sequence, depending on $r_{w_{-k}}$. Assume first $r_{w_{-k}} \in X$, which is guaranteed in case $k < n$ by the maximality of k. Suppose that $\widetilde{T}^{k+1} r_{w_{-k}} \ne T^{k+1} r_{w_{-k}}$, and let i, $m_{w_{-k}} \le i \le k$, be maximal such that $\widetilde{T}^i r_{w_{-k}} = T^i r_{w_{-k}}$. Since $\widetilde{T}^{i+1} r_{w_{-k}} \ne T^{i+1} r_{w_{-k}}$, we have $\widetilde{T}^i r_{w_{-k}} = r_{w_{-k+i}} \in X$. Furthermore, the maximality of i implies $m_{w_{-k+i}} > k-i+1$ and thus $\widetilde{T}^{k+1} r_{w_{-k}} = \widetilde{T}^{k-i+1} r_{w_{-k+i}} \in \mathcal{V}$, contradicting that $(x, y\,] \cap \mathcal{V} = \emptyset$. Therefore we have $\widetilde{T}^{k+1} r_{w_{-k}} = T^{k+1} r_{w_{-k}}$, hence Lemma 5.3.8 gives

$$\varphi(w_{-k} \cdots w_0) = \varphi\big(b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}})\big).$$

Next we show that $w_{-n} \cdots w_{-k-1} b(r_{w_{-k}})$ is admissible. We clearly have for $k < i \le n$ that

$$w_{-i} \cdots w_{-k-1} b(r_{w_{-k}}) \succ w_{-i} \cdots w_0 b(x) \succeq b(\ell_{w_{-i}}).$$

Suppose that $w_{-i} \cdots w_{-k-1} b(r_{w_{-k}}) \succeq \tilde{b}(r_{w_{-i}})$. Since $w_{-i} \cdots w_0 b(x) \prec \tilde{b}(r_{w_{-i}})$, it follows that $w_{-i} \cdots w_{-k-1} = \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i-k}(r_{w_{-i}})$, hence

$$b(r_{w_{-k}}) \succeq \tilde{b}(\widetilde{T}^{i-k} r_{w_{-i}}) \succ w_{-k} \cdots w_0 b(x) = \tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}) b(x).$$

Since $b(r_{w_{-k}}) \succeq \tilde{b}(\widetilde{T}^{i-k} r_{w_{-i}})$ implies $r_{w_{-k}} \ge \widetilde{T}^{i-k} r_{w_{-i}}$ and thus $\tilde{b}(r_{w_{-k}}) \succeq \tilde{b}(\widetilde{T}^{i-k} r_{w_{-i}})$, we obtain $\tilde{b}_1(r_{w_{-k}}) \cdots \tilde{b}_{k+1}(r_{w_{-k}}) = \tilde{b}_{i-k+1}(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$, hence $w_{-i} \cdots w_0 = \tilde{b}_1(r_{w_{-i}}) \cdots \tilde{b}_{i+1}(r_{w_{-i}})$, contradicting the maximality of k. This shows that $w_{-n} \cdots w_{-k-1} b(r_{w_{-k}})$ is admissible. If

$$w_{-n} \cdots w_{-k-1} b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}}) b(y)$$

is admissible as well, then we can take

$$w'_{-n} \cdots w'_0 = w_{-n} \cdots w_{-k-1} b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}}),$$

and (5.11) holds with vanishing error term.

Now assume that $r_{w_{-k}} \notin X$, which implies $k = n$. Note that $\tilde{b}(r_{w_{-k}})$ cannot be purely periodic since this would imply $m_{w_{-k}} = \infty$ and contradict

$(x, y] \cap \mathcal{V} = \emptyset$. By Lemma 5.3.7, there exists therefore some h and some $a \in A$ such that $r_a \in X$ and $\widetilde{T}^h r_a = r_{w_{-n}}$. As above, we obtain

$$\varphi\big(\tilde{b}_1(r_a) \cdots \tilde{b}_h(r_a) w_{-n} \cdots w_0\big) = \varphi\big(b_1(r_a) \cdots b_{n+h+1}(r_a)\big),$$

and thus

$$\varphi(w_{-n} \cdots w_0) = \varphi\big(b_{h+1}(r_a) \cdots b_{n+h+1}(r_a)\big) + \mathcal{O}(\rho^n).$$

If $b_{h+1}(r_a) \cdots b_{n+h+1}(r_a) b(y)$ is admissible, then we are done.

If $w_{-n} \cdots w_{-k-1} b_1(r_{w_{-k}}) \cdots b_{k+1}(r_{w_{-k}})$ or $b_{h+1}(r_a) \cdots b_{n+h+1}(r_a)$ respectively is not the desired sequence, then we iterate the process with this sequence as new $w_{-n} \cdots w_0$, and $\widetilde{T}^{k+1} r_{w_{-k}}$ as new x. Since these numbers $\widetilde{T}^{k+1} r_{w_{-k}}$ increase and can take only finitely many values, the algorithm terminates. The number of instances of $k = n$ is bounded by the size of A, thus the error term only depends on T and is $\mathcal{O}(\rho^n)$. $\square$

Remark 5.3.1. We can prove Lemma 5.3.9 also in the other direction: Suppose that there is a finite sequence $w_{-n} \cdots w_0$, such that $w_{-n} \cdots w_0 b(y)$ is admissible, but $w_{-n} \cdots w_0 b(x)$ is not. Then, there is another sequence $w'_{-n} \cdots w'_0$, such that $w'_{-n} \cdots w'_0 b(x)$ is admissible and

$$\varphi(w'_{-n} \cdots w'_0) = \varphi(w_{-n} \cdots w_0) + \mathcal{O}(\rho^n).$$

The proof is done in exactly the same way.

Proof of Proposition 5.3.1. Take $x, y \in X$, that satisfy the conditions of the proposition, and assume that there exists $w = (w_k)_{k \le 0}$ such that $w \cdot b(x) \in \mathcal{S}$, but $w \cdot b(y) \notin \mathcal{S}$. For each $n \ge 0$, let $w_{-n}^{(n)} \cdots w_0^{(n)}$ be the sequence given by Lemma 5.3.9. By the surjectivity of T, we can extend this sequence to a sequence $w^{(n)} = (w_k^{(n)})_{k \le 0}$ with $w^{(n)} \cdot b(y) \in \mathcal{S}$ satisfying $\varphi(w^{(n)}) = \varphi(w) + \mathcal{O}(\rho^n)$. Hence, $\lim_{n \to \infty} \varphi(w^{(n)}) = \varphi(w)$. Since $\varphi(w^{(n)}) \in \mathcal{D}_y$ for each $n \ge 0$ and $\mathcal{D}_y$ is closed, we obtain that $\varphi(w) \in \mathcal{D}_y$. By Remark 5.3.1, we also obtain that $\varphi(w) \in \mathcal{D}_x$ if $\varphi(w) \in \mathcal{D}_y$, and the proposition is proved. $\square$

Suppose that $\mathcal{V}$ is a finite set. From Proposition 5.3.1 it follows that we then have only finitely many different sets $\mathcal{D}_x$. For every $i \in I$, let the elements of $\mathcal{V} \cap X_i$ be given by

$$\ell_{i,0} < \ell_{i,1} < \cdots < \ell_{i,N_i},$$

and the elements of $\mathcal{V} \cap \widetilde{X}_i$ be

$$r_{i,0} < r_{i,1} < \cdots < r_{i,N_i},$$

i.e., $\ell_{i,0} = \ell_i$, $r_{i,n} = \ell_{i,n+1}$ for $0 \le n < N_i$, $r_{i,N_i} = r_i$. Let $X_{i,n} = [\ell_{i,n}, r_{i,n})$. Then we have

$$\widehat{X} = \bigcup_{i \in I} \bigcup_{0 \le n \le N_i} \widehat{X}_{i,n} \quad \text{with} \quad \widehat{X}_{i,n} = X_{i,n} \mathbf{v}_1 - \mathcal{D}_{\ell_{i,n}}.$$

Use $\mathcal{D}_{i,n}$ to denote $\mathcal{D}_{\ell_{i,n}}$. The next proposition generalizes Theorem 5.1.5 of Akiyama. We need the following lemma.

Lemma 5.3.10. *Suppose that $\mathcal{V}$ is finite. For every $x \in X$ we have*

$$\mathcal{D}_x = \bigcup_{y \in T^{-1}\{x\}} \Big(M_\beta \mathcal{D}_y + \varphi\big(b_1(y)\big) \Big), \tag{5.12}$$

where the union is disjoint up to sets of measure zero.

Proof. Let $x \in X$. Then, for every $i \in I$ such that $ib(x)$ is admissible, there is a unique $y \in T^{-1}\{x\}$, such that $b(y) = ib(x)$. Moreover, for each $w \in {}^\omega I$ we have $wi \cdot b(x) \in \mathcal{S}$ if and only if $w \cdot ib(x) \in \mathcal{S}$. Since $\varphi(wi) = M_\beta \varphi(w) + \varphi(i)$, we have (5.12). Let $y \in T^{-1}\{x\}$ be in the interval $X_{b_1(y),n}$ and $y' \in T^{-1}\{x\}$, $y' \neq y$, be in $X_{b_1(y'),n'}$, then

$$\begin{aligned}&\widehat{T}\widehat{X}_{b_1(y),n} \cap \widehat{T}\widehat{X}_{b_1(y'),n'} \\ &\qquad = \big(TX_{b_1(y),n} \cap TX_{b_1(y'),n'}\big)\mathbf{v}_1 + \big(M_\beta \mathcal{D}_y + \varphi(b_1(y))\big) \cap \big(M_\beta \mathcal{D}_{y'} + \varphi(b_1(y'))\big).\end{aligned}$$

Since this set is contained in $\widehat{T}\widehat{X}_{b_1(y)} \cap \widehat{T}\widehat{X}_{b_1(y')}$ and $b_1(y) \neq b_1(y')$, it has zero Lebesgue measure by Lemma 5.3.3. Since $TX_{b_1(y),n} \cap TX_{b_1(y'),n'}$ is a non-empty left-closed and right-open interval, this implies that the union in (5.12) is disjoint up to a set of measure zero. □

Proposition 5.3.2. *If $\mathcal{V}$ is finite, then $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$ for all $x \in X$.*

Proof. If $\lambda^{d-1}(\mathcal{D}_x) = 0$, then $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$ since $\mathcal{D}_x$ is compact. Therefore we can assume $\lambda^{d-1}(\mathcal{D}_x) > 0$. We first show that there is a $y \in X$ with $\lambda^{d-1}(\mathcal{D}_y) > 0$ and $\lambda^{d-1}(\partial \mathcal{D}_y) = 0$ and then extend this to arbitrary $x \in X$.

By Lemma 5.3.5 we know that $\mathbb{R}^d = \bigcup_{\mathbf{z} \in \mathbb{Z}^d} (\mathbf{z} + \widehat{Y})$. Then by Baire's Theorem and the fact that $\widehat{Y}$ is compact, $\widehat{Y}$ contains an open ball. This implies that there is a $z \in \mathbb{Z}[\beta]$, such that $\mathcal{D}_z$ contains an inner point with respect to H. By iterating (5.12), we obtain for all $k \geq 1$ that $\mathcal{D}_z$ is the (up to a set of measure zero) disjoint union of $M_\beta^k \mathcal{D}_y + \varphi\big(b_1(y) \cdots b_k(y)\big)$, $y \in T^{-k}\{z\}$. Since M_β is contracting on H, there must be some $y \in T^{-k}\{z\}$ for sufficiently large k such that $M_\beta^k \mathcal{D}_y + \varphi\big(b_1(y) \cdots b_k(y)\big)$ is contained in the interior of $\mathcal{D}_z$ (with respect to H). Then for every point in $\partial\Big(M_\beta^k \mathcal{D}_y + \varphi\big(b_1(y) \cdots b_k(y)\big)\Big)$ there is a $y' \in T^{-k}\{z\}$, $y' \neq y$, such that it also lies in $M_\beta^k \mathcal{D}_{y'} + \varphi\big(b_1(y') \cdots b_k(y')\big)$. Since the intersection of these sets has zero measure, we obtain $\lambda^{d-1}(\partial \mathcal{D}_y) = 0$.

Now, consider a set $M_\beta \mathcal{D}_{z'} + \varphi\big(b_1(z')\big)$, $z' \in T^{-1}\{y\}$. Then every points from the boundary of this set is either also in the boundary of another set from the union in (5.12) or not. If not, then it is in $\partial \mathcal{D}_y$. Therefore, we have $\lambda^{d-1}(\partial \mathcal{D}_{z'}) = 0$ for all $z' \in T^{-1}\{y\}$. It remains to show that, when iterating this argument, every $\mathcal{D}_x$ with $\lambda^{d-1}(\mathcal{D}_x) > 0$ occurs eventually in one of these subdivisions. Since by Proposition 5.3.1, $\mathcal{D}_x$ does not change within $X_{i,n}$, we set

$$\mathcal{N} = \big\{(i,n) \,|\, \lambda^{d-1}(\mathcal{D}_{i,n}) > 0\big\} = \big\{(i,n) \,|\, \lambda^{d}(\widehat{X}_{i,n}) > 0\big\}.$$

Let $\mathcal{G}$ be the weighted oriented graph with set of nodes $\mathcal{N}$ and an edge from (i,n) to (i',n') if and only if $T^{-1}X_{i,n} \cap X_{i',n'} \neq \emptyset$. Then we have $\lambda^{d-1}(\partial \mathcal{D}_x) = 0$

for every $x \in X_{i',n'}$ if (i',n') can be reached from the node (i,n) satisfying $y \in X_{i,n}$.

Let the weight of an edge be $\lambda^d(\widehat{T}^{-1}\widehat{X}_{i,n} \cap \widehat{X}_{i',n'}) > 0$. Since $\widehat{T}$ is bijective off of a set of Lebesgue measure zero, the sum of the weights of the outgoing edges as well as the sum of the weights of the ingoing edges must equal $\lambda^d(\widehat{X}_{i,n})$ for every node $(i,n) \in \mathcal{N}$. This implies that every connected component of $\mathcal{G}$ must be strongly connected, in the sense that if two nodes are in the same connected component, then there is a path from one node to the other and the other way around. By the definition of $\mathcal{G}$, the restriction of T to a strongly connected component is a right-continuous β-transformation, and this restriction changes the tiles in this component only by sets of measure zero. Therefore, the previous arguments provide some $y \in X$ with $\lambda^{d-1}(\mathcal{D}_y) > 0$ and $\lambda^{d-1}(\partial\mathcal{D}_y) = 0$ in every connected component of $\mathcal{G}$. Hence, we obtain $\lambda^{d-1}(\partial D_x) = 0$ for every $x \in X$. $\square$

Now, consider the measure μ, defined by $\mu(E) = (\lambda^d \circ \pi^{-1})(E)$ for all measurable sets E. If we take an interval $[s,t) \subseteq X$, then there exists some constant $c > 0$, such that

$$
\begin{aligned}
\mu([s,t)) &= (\lambda^d \circ \pi^{-1})\Big(\bigcup_{i\in I}\bigcup_{n=0}^{N_i} X_{i,n} \cap [s,t)\Big) = \sum_{i,n} c\,\lambda(X_{i,n} \cap [s,t))\,\lambda^{d-1}(\mathcal{D}_{i,n}) \\
&= c\int_{[s,t)} \sum_{i,n} \lambda^{d-1}(\mathcal{D}_{i,n})\,1_{X_{i,n}}\,d\lambda.
\end{aligned}
$$

Hence, the support of μ is the union of intervals $X_{i,n}$, such that $\lambda^{d-1}(\mathcal{D}_{i,n}) > 0$. Since $\lambda^d(\widehat{X}) > 0$, the support of μ has positive Lebesgue measure. Hence, μ is absolutely continuous with respect to the Lebesgue measure on X. If the transformation T has a unique invariant measure, that is absolutely continuous with respect to the Lebesgue measure, then μ must be this measure. This is the case for the classical greedy and lazy β-transformations and the greedy and lazy β-transformations with arbitrary digits.

5.3.3 *Examples of natural extensions*

Example 5.3.1. Let β be the golden mean, i.e., the positive solution of the equation $x^2 - x - 1 = 0$. The other solution of this equation is $\beta_2 = -1/\beta$. Then

$$
M_\beta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_1 = \frac{1}{\beta^2+1}\begin{pmatrix} \beta^2 \\ \beta \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\beta^2+1}\begin{pmatrix} 1 \\ -\beta \end{pmatrix}.
$$

The classical greedy β-transformation, T_β, is given by $I = A = \{0,1\}$, $X_0 = [0,1/\beta)$ and $X_1 = [1/\beta,1)$. We have $T_\beta\ell_0 = T_\beta\ell_1 = \ell_0 = 0$, $T_\beta r_0 = r_1 = 1$ and $\widetilde{T}_\beta r_1 = r_0 = \frac{1}{\beta}$. Thus, $\mathcal{V} = \{0, \frac{1}{\beta}, 1\}$. The transformation and its natural extension are depicted in Figure 5.2.

Example 5.3.2. Let β be the golden mean again and now take $I = A = \{-1,1\}$. We will use $\bar{1}$ to denote -1, whenever that is more convenient. Set $X_{\bar{1}} =$

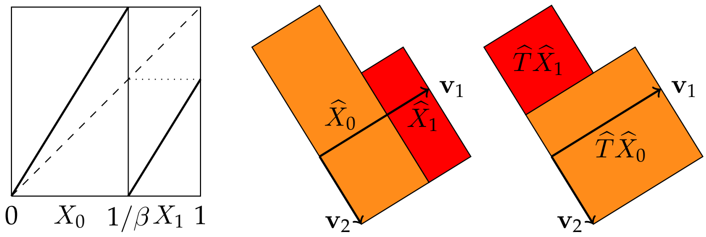


Figure 5.2 The classical greedy β-transformation T_β for $\beta = \frac{1+\sqrt{5}}{2}$, and its natural extension.

$[-1, 0)$, $X_1 = [0, 1)$. Then $T^3\ell_{\bar{1}} = T^2(-1/\beta) = T\,0 = T\ell_1 = -1 = \ell_{\bar{1}}$ and $\widetilde{T}^3 r_{\bar{1}} = \widetilde{T}^2 1 = \widetilde{T}^2 r_1 = \widetilde{T}(1/\beta) = 0 = r_{\bar{1}}$. Thus, $\mathcal{V} = \{-1, -\frac{1}{\beta}, 0, \frac{1}{\beta}, 1\}$. The transformation and its natural extension are depicted in Figure 5.3.

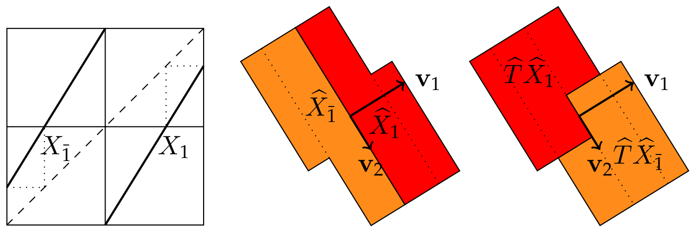


Figure 5.3 The transformation from Example 5.3.2 and its natural extension.

Example 5.3.3. Let β be the golden mean again, but $I = A = \{-1, 0, 1\}$. Set

$$X_{\bar{1}} = \left[-\frac{\beta^2+\beta^{-3}}{\beta^2+1}, -\frac{\beta+\beta^{-4}}{\beta^2+1}\right), \quad X_0 = \left[-\frac{\beta+\beta^{-4}}{\beta^2+1}, \frac{\beta+\beta^{-4}}{\beta^2+1}\right),$$

$$X_1 = \left[\frac{\beta+\beta^{-4}}{\beta^2+1}, \frac{\beta^2+\beta^{-3}}{\beta^2+1}\right).$$

The endpoints have the expansions

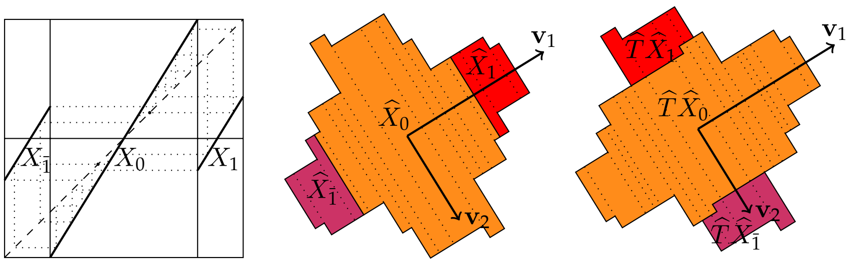


Figure 5.4 The transformation from Example 5.3.3 and its natural extension.

$$b(\ell_{\bar{1}}) = \bar{1}(00\bar{1}0)^\omega,\ \tilde{b}(r_{\bar{1}}) = \bar{1}0010(000\bar{1})^\omega,\ b(\ell_0) = b(r_{\bar{1}}) = 0\bar{1}(00\bar{1}0)^\omega,$$
$$\tilde{b}(r_0) = 01(0010)^\omega,\ b(\ell_1) = b(r_0) = 100\bar{1}0(0001)^\omega,\ \tilde{b}(r_1) = 1(0010)^\omega.$$

Since $\widetilde{T}^5 r_{\bar{1}} = T^5 r_{\bar{1}}$ and $\widetilde{T}^5 r_0 = T^5 r_0$, we have $m_{\bar{1}} = m_0 = 5$, and thus

$$\begin{aligned}\mathcal{V} = \big\{&.\bar{1}(00\bar{1}0)^\omega, .(\bar{1}000)^\omega, .\bar{1}0(0001)^\omega, .\bar{1}0010(000\bar{1})^\omega = .0\bar{1}(00\bar{1}0)^\omega,\\ &.(0\bar{1}00)^\omega, .0\bar{1}0(0001)^\omega, .(00\bar{1}0)^\omega, .00\bar{1}0(0001)^\omega, .0(000\bar{1})^\omega\\ &.0(0001)^\omega, .0010(000\bar{1})^\omega, .(0010)^\omega, .010(000\bar{1})^\omega, .(0100)^\omega,\\ &.01(0010)^\omega = .100\bar{1}0(0001)^\omega, .10(000\bar{1})^\omega, .(1000)^\omega, .1(0010)^\omega\big\}.\end{aligned}$$

The transformation and its natural extension are depicted in Figure 5.4. The dotted lines in the natural extension separate the different intervals $X_{i,n}$. At the point $.(0001)^\omega$ we do not have a dotted line, but the set $\widehat{T}\widehat{X}_1$ changes its shape. This implies that there the set $\mathcal{D}_x$ does not change, but the decomposition according to (5.12) changes. The same holds for the point $.(000\bar{1})^\omega$ and the set $\widehat{T}\widehat{X}_{\bar{1}}$.

Example 5.3.4. Let β be the golden mean again and $I = A = \{-1, 0, 1\}$, but $X_{\bar{1}} = \big[-\frac{1}{2}, -\frac{1}{2\beta}\big)$, $X_0 = \big[-\frac{1}{2\beta}, \frac{1}{2\beta}\big)$ $X_1 = \big[\frac{1}{2\beta}, \frac{1}{2}\big)$. This is an example of a symmetric β-transformation defined by Akiyama and Scheicher ([AS07]), see also Example 5.2.5. The endpoints have the expansions

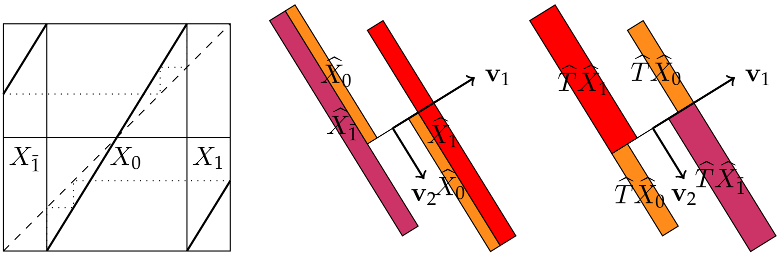


Figure 5.5 The transformation from Example 5.3.4 and its natural extension.

$$b(\ell_{\bar{1}}) = (\bar{1}01)^\omega,\ \tilde{b}(r_{\bar{1}}) = (\bar{1}10)^\omega,\ b(\ell_0) = b(r_{\bar{1}}) = 0(\bar{1}01)^\omega,$$
$$\tilde{b}(r_0) = 0(10\bar{1})^\omega,\ b(\ell_1) = b(r_0) = (1\bar{1}0)^\omega,\ \tilde{b}(r_1) = (10\bar{1})^\omega.$$

Thus, $\mathcal{V} = \big\{-\frac{1}{2}, -\frac{1}{2\beta}, -\frac{1}{2\beta^2}, \frac{1}{2\beta^2}, \frac{1}{2\beta}, \frac{1}{2}\big\}$. The transformation and its natural extension are depicted in Figure 5.5.

Example 5.3.5. Consider the Tribonacci number, i.e., let β be the real solution of the equation $x^3 - x^2 - x - 1 = 0$. Let $I = A = \{-1, 0, 1\}$ and take $X_{\bar{1}} = \big[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\big)$, $X_0 = \big[-\frac{1}{\beta+1}, \frac{1}{\beta+1}\big)$, $X_1 = \big[\frac{1}{\beta+1}, \frac{\beta}{\beta+1}\big)$. This is a minimal weight transformation as defined by Frougny and Steiner in [FS08], see also Example 5.2.3. The endpoints have the following expansions.

$$b(\ell_{\bar{1}}) = (\bar{1}00)^\omega,\ b(\ell_0) = (0\bar{1}0)^\omega,\ b(\ell_1) = 1(0\bar{1}0)^\omega,$$
$$\tilde{b}(r_{\bar{1}}) = \bar{1}(010)^\omega,\ \tilde{b}(r_0) = (010)^\omega,\ \tilde{b}(r_1) = (100)^\omega,$$

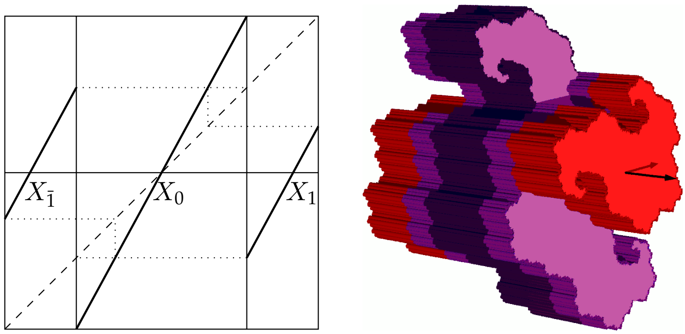


Figure 5.6 The transformation of Example 5.3.5 and the natural extension domain $\widehat{X}$.

and $\mathcal{V} = \big\{ -\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}, -\frac{1}{\beta(\beta+1)}, \frac{1}{\beta(\beta+1)}, \frac{1}{\beta+1}, \frac{\beta}{\beta+1} \big\}$. We see the transformation and the region $\widehat{X}$ in Figure 5.6.

Example 5.3.6. Consider the smallest Pisot number, i.e., let β be the real solution of the equation $x^3 - x - 1 = 0$. Let $I = A = \{-1, 0, 1\}$ and take $X_{\bar{1}} = \big[-\frac{\beta^7}{\beta^8-1}, -\frac{\beta^6}{\beta^8-1} \big)$, $X_0 = \big[-\frac{\beta^6}{\beta^8-1}, \frac{\beta^6}{\beta^8-1} \big)$, $X_1 = \big[\frac{\beta^6}{\beta^8-1}, \frac{\beta^7}{\beta^8-1} \big)$. This is also a minimal weight transformation as defined in [FS08]. The endpoints have the following expansions.

$$b(\ell_{\bar{1}}) = (\bar{1}0^7)^\omega,\ b(\ell_0) = (0\bar{1}0^6)^\omega,\ b(\ell_1) = 1(0^6\bar{1}0)^\omega,$$
$$\tilde{b}(r_{\bar{1}}) = \bar{1}(0^6 10)^\omega,\ \tilde{b}(r_0) = (010^6)^\omega,\ \tilde{b}(r_1) = (10^7)^\omega,$$

and $\mathcal{V} = \big\{ \frac{\pm\beta^k}{\beta^8-1} \,|\, 0 \le k \le 7 \big\}$. We see the transformation and the region $\widehat{X}$ in Figure 5.7.

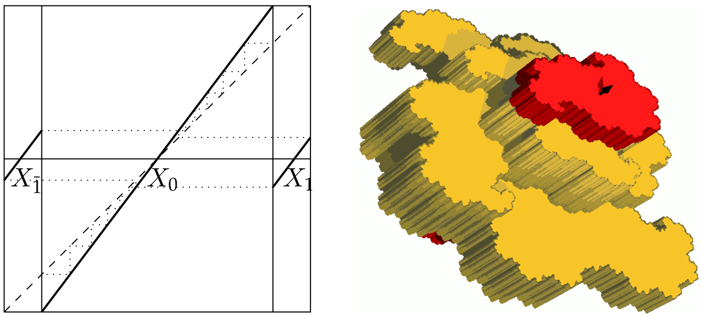


Figure 5.7 The transformation of Example 5.3.6 and the natural extension domain $\widehat{X}$.

5.4 Multiple tilings

In this section, we consider a right-continuous β-transformation $T : X \to X$ as in Definition 5.2.3 satisfying all assumptions from Section 5.3, i.e., that β is a Pisot unit, $A \subset \mathbb{Q}(\beta)$, $TX = X$ and $\mathcal{V}$ is finite.

If Z denotes the support of the invariant measure $\mu = \lambda^d \circ \pi^{-1}$, then we have $\lambda^{d-1}(\mathcal{D}_x) > 0$ if $x \in Z$, $\lambda^{d-1}(\mathcal{D}_x) = 0$ if $x \in X \backslash Z$ and $TZ = Z$. Therefore, the sets $\mathcal{D}_x$ defined by $T|_Z$ differ from those defined by T only by sets of measure zero. In order to avoid having sets $\mathcal{D}_x$ of measure zero, we assume that X is the support of the invariant measure $\mu = \lambda^d \circ \pi^{-1}$.

The next lemma says that every right-continuous β-transformation with digits in $\mathbb{Q}(\beta)$ is isomorphic to a right-continuous β-transformation with digits in $\mathbb{Z}[\beta]$. Therefore we can assume, w.l.o.g., that $A \subset \mathbb{Z}[\beta]$.

Lemma 5.4.1. *Let T be a right-continuous β-transformation with $A \subset \mathbb{Q}(\beta)$. Let $q \in \mathbb{Z}\backslash\{0\}$ be such that $A \subset q^{-1}\mathbb{Z}[\beta]$. Define another right-continuous β-transformation S by $I_S = I$, $a_i^S = q\,a_i$ for all $i \in I$ and $X_i^S = q\,X_i$. Hence $X^S = q\,X$. Then the function $\theta : X \to X^S : x \mapsto q\,x$ is a continuous bijection with $\theta \circ T = S \circ \theta$.*

Proof. It is obvious that θ is a continuous bijection. To show that $\theta \circ T = S \circ \theta$, take $x \in X_i$. Then $\theta(Tx) = q(\beta x - a_i)$. On the other hand, $\theta(x) \in X_i^S$, so $S\theta(x) = \beta(qx) - a_i^S = q\,\beta x - q\,a_i$. □

5.4.1 *Tiling of the contracting hyperplane*

For $x \in \mathbb{Q}(\beta)$, let

$$\Xi(x) = \sum_{j=2}^{d} \Gamma_j(x)\mathbf{v}_j \in H.$$

We define tiles in the hyperplane H by $\mathcal{T}_x = \Xi(x) + \mathcal{D}_x$ for $x \in \mathbb{Q}(\beta) \cap X$. However, we will usually restrict x to $\mathbb{Z}[\beta] \cap X$ in order to obtain a (multiple) tiling. By Lemma 5.3.6, Proposition 5.3.1 and Proposition 5.3.2, these tiles are translates of a finite collection of compact sets, the boundary of which has zero Lebesgue measure. Observe that for $x \in \mathbb{Q}(\beta)$ and $w \in {}^{\omega}I$ we have

$$M_\beta(\Xi(x) + \varphi(w)) = \Xi(Tx) + \varphi(wb_1(x)).$$

To show that the family

$$\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X} = \{\Xi(x) + \mathcal{D}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$$

is a multiple tiling of the space H, we have to show (mt1)-(mt4) for some $m \geq 1$. We begin by proving that the set of translation vectors, $\Xi(\mathbb{Z}[\beta] \cap X)$, is a model set of a cut and project scheme and then use Theorem 5.1.2 to obtain that the set $\Xi(\mathbb{Z}[\beta] \cap X)$ is a Delone set, i.e., that it is uniformly discrete and relatively dense, see Definition 5.1.4 and Definition 5.1.3. We also need the following lemma.

Lemma 5.4.2. *For every $\mathbf{x} \in \mathbb{Q}^d$, there is a unique $x \in \mathbb{Q}(\beta)$, such that $\mathbf{x} = \Xi(x) + x\mathbf{v}_1$. If $\mathbf{x} \in \mathbb{Z}^d$, then $x \in \mathbb{Z}[\beta]$. The map $\Xi : \mathbb{Q}(\beta) \to H$ is injective.*

Proof. By the structure of M_β we can write every $\mathbf{x} \in \mathbb{Q}^d$ in a unique way as $\mathbf{x} = \sum_{k=0}^{d-1} x_k M_\beta^k (1, 0, \ldots, 0)^t$ with $x_k \in \mathbb{Q}$ for all $0 \leq k \leq d-1$. From this we have

$$\mathbf{x} = \sum_{k=0}^{d-1} x_k \sum_{j=1}^{d} \beta_j^k \mathbf{v}_j = \Xi\Big(\sum_{k=0}^{d-1} x_k\beta^k\Big) + \Big(\sum_{k=0}^{d-1} x_k\beta^k\Big)\mathbf{v}_1.$$

This proved the first part of the lemma. For the second part, we just note that the elements x_k are in $\mathbb{Z}$, if $\mathbf{x} \in \mathbb{Z}^d$. For the last part, note that we also have that for each $q \in \mathbb{Q}(\beta)$, $\mathbf{q} = \Xi(q)+q\mathbf{v}_1 \in \mathbb{Q}^d$. Now, for $q, q' \in \mathbb{Q}(\beta)$ with $q \neq q'$ we have $\Xi(q) - \Xi(q') = (\mathbf{q} - \mathbf{q}') + (q - q')\mathbf{v}_1$. Since the coefficients of $\mathbf{v}_1$ are linearly independent over $\mathbb{Q}$, we have $(q - q')\mathbf{v}_1 \notin \mathbb{Q}^d$ and thus $\Xi(q) - \Xi(q') \neq \mathbf{0}$. □

Note that from the proof of the previous lemma it follows that the map $x \mapsto \Xi(x) + x\mathbf{v}_1$ is a bijection from $\mathbb{Q}(\beta)$ to $\mathbb{Q}^d$.

Lemma 5.4.3. *Each set $\Xi(q^{-1}\mathbb{Z}[\beta] \cap E)$, where $q \in \mathbb{Z} \setminus \{0\}$ and $E \subset \mathbb{R}$ is bounded and has non-empty interior, is uniformly discrete and relatively dense. In particular, this holds for $\Xi(\mathbb{Z}[\beta] \cap X)$ and all the sets $\Xi(\mathbb{Z}[\beta] \cap X_{i,n})$, $i \in I$, $0 \leq n \leq N_i$.*

Proof. Write $\mathbf{x} \in \mathbb{R}^d$ as $\mathbf{x} = x\mathbf{v}_1 + \mathbf{y}$ with $x \in \mathbb{R}$ and $\mathbf{y} \in H$. Define two projections $\pi_1 : \mathbb{R}^d \to \mathbb{R}$ and $\pi_H : \mathbb{R}^d \to H$ by $\pi_1(\mathbf{x}) = x$ and $\pi_H(\mathbf{x}) = \mathbf{y}$ and set $\iota(\mathbf{x}) = (\pi_H(\mathbf{x}), \pi_1(\mathbf{x}))$. We first show that $(H \times \mathbb{R}\mathbf{v}_1, \iota(q^{-1}\mathbb{Z}^d))$ is a cut and project scheme. Since $H \simeq \mathbb{R}^{d-1}$ and $\iota(q^{-1}\mathbb{Z}^d)$ is a lattice, we only need to show that π_H is injective on $\mathbb{Z}^d$ and that $\pi_1(\mathbb{Z}^d)$ is dense in $\mathbb{R}$. By Lemma 5.4.2 we know that for each $\mathbf{z} \in q^{-1}\mathbb{Z}^d$, there is a unique element $z \in q^{-1}\mathbb{Z}[\beta]$ with $\mathbf{z} = \Xi(z) + z\mathbf{v}_1$. Then $\pi_1(\mathbf{z}) = z$ and $\pi_H(\mathbf{z}) = \Xi(z)$. Moreover, by Lemma 5.4.2 we have for each $z \in q^{-1}\mathbb{Z}[\beta]$, that $\mathbf{z} = \Xi(z) + z\mathbf{v}_1 \in q^{-1}\mathbb{Z}^d$. Hence, $\pi_1(q^{-1}\mathbb{Z}^d) = q^{-1}\mathbb{Z}[\beta]$, which is dense in $\mathbb{R}$. The injectivity of π_H on $\mathbb{Z}^d$ follows from the last part of Lemma 5.4.2. Hence, $(H \times \mathbb{R}\mathbf{v}_1, \iota(q^{-1}\mathbb{Z}^d))$ is a cut and project scheme.

Now, let $E \subset \mathbb{R}$ be a bounded set with non-empty interior. Then the set

$$\Xi(q^{-1}\mathbb{Z}[\beta] \cap E) = \{\pi_H(\mathbf{z}) \mid \mathbf{z} \in q^{-1}\mathbb{Z}^d,\ \pi_1(\mathbf{z}) \in E\}$$

is a model set, and therefore a Delone set by Theorem 5.1.2. □

As an immediate corollary we have that the family $\mathcal{T}$ is locally finite, see Definition 5.1.2 for the definition. Hence, we have (mt2).

Corollary 5.4.1. *The family $\mathcal{T} = \{\mathcal{T}_x\}_{x\in\mathbb{Z}[\beta]\cap X}$ is locally finite.*

Proof. By the uniform discreteness of the set $\Xi(\mathbb{Z}[\beta] \cap X)$, each ball in H contains only finitely many elements from $\Xi(\mathbb{Z}[\beta] \cap X)$. Since the sets $\mathcal{D}_x$ are bounded, the number of elements in the set $\{x \in \mathbb{Z}[\beta] \cap X \mid \mathcal{T}_x \cap B(\mathbf{y}, r) \neq \emptyset\}$ is finite. By the injectivity of Ξ we have the lemma. □

The next lemma states that the collection $\mathcal{T}$ covers the whole space H, i.e., it gives (mt3).

Lemma 5.4.4. $H = \bigcup_{x\in\mathbb{Z}[\beta]\cap X} \mathcal{T}_x$.

Proof. Let $J = \bigcup_{x \in \mathbb{Z}[\beta] \cap X} \mathcal{T}_x$. Obviously $J \subseteq H$. For the converse, note that every point $\mathbf{y} \in J$ can be written as $\mathbf{y} = \varphi(w) + \Xi(x)$, with $w \in {}^{\omega}I$, $x \in \mathbb{Z}[\beta] \cap X$, $w \cdot b(x) \in \mathcal{S}$. Then we have $M_\beta \mathbf{y} = \varphi(w b_1(x)) + \Xi(Tx)$. Since $a_{b_1(x)} \in \mathbb{Z}[\beta]$, we have $Tx \in \mathbb{Z}[\beta] \cap X$ and thus $M_\beta \mathbf{y} \in J$. Hence, $M_\beta J \subseteq J$ Since every tile is non-empty and $\Xi(\mathbb{Z}[\beta] \cap X)$ is relatively dense in H, J is also relatively dense. Since $M_\beta J \subseteq J$, this implies that J is dense in H. Since the tiles are compact and $\mathcal{T}$ is locally finite, we obtain $J = H$. □

We now prove (mt1), i.e., we prove that the tiles are the closure of their interior. For a subset $E \subseteq H$, let $\mathrm{int}(E)$ denote the interior of E and $\overline{E}$ the closure.

Lemma 5.4.5. *Let $\mathcal{T}_x \in \mathcal{T}$. Then $\mathcal{D}_x = \overline{\mathrm{int}(\mathcal{D}_x)}$.*

Proof. Since the tiles are closed, we have $\overline{\mathrm{int}(\mathcal{D}_x)} \subseteq \mathcal{D}_x$. For the other inclusion, we show $\mathcal{T}_x \subseteq \overline{\mathrm{int}(\mathcal{T}_x)}$. Let $\mathbf{y} \in \mathcal{T}_x$ for some $x \in \mathbb{Z}[\beta] \cap X$. Then $\mathbf{y} = \varphi(w) + \Xi(x)$ for some $w = (w_k)_{k \leq 0} \in {}^{\omega}I$ with $w \cdot b(x) \in \mathcal{S}$. Set $x_k = \bullet w_{-k+1} \cdots w_0 b(x)$ for $k \geq 0$. Then we have $M_\beta^k\big(\Xi(x_k) + \mathcal{D}_{x_k}\big) \subseteq \mathcal{T}_x$. Since $\lambda^{d-1}(\mathcal{D}_{x_k}) > 0$ and $\lambda^{d-1}(\partial \mathcal{D}_{x_k}) = 0$, there exists a point $\mathbf{y}_k \in M_\beta^k\big(\Xi(x_k) + \mathrm{int}(\mathcal{D}_{x_k})\big) \subseteq \mathrm{int}(\mathcal{T}_x)$. Then $\mathbf{y}_k = \varphi(w^{(k)})$ for a sequence $w^{(k)} = (w_n^{(k)})_{n \leq 0} \in {}^{\omega}I$ with $w^{(k)} \cdot b(x) \in \mathcal{S}$ and $w_{-k+1}^{(k)} \cdots w_0^{(k)} = w_{-k+1} \cdots w_0$. Since $\lim_{k \to \infty} \mathbf{y}_k = \mathbf{y}$, we have $\mathbf{y} \in \overline{\mathrm{int}(\mathcal{T}_x)}$. □

The matrix M_β is contracting on H. Hence M_β^{-1} is expanding. For each element $\mathcal{T}_x$ from $\mathcal{T}$, we can write $M_\beta^{-1}\mathcal{T}_x$ as the union of elements from $\mathcal{T}$, according to (5.12). We have for each $x \in \mathbb{Z}[\beta] \cap X$, that

$$\begin{aligned} M_\beta^{-1}\mathcal{T}_x &= M_\beta^{-1}\Big(\Xi(x) + \bigcup_{y \in T^{-1}\{x\}} \big(M_\beta \mathcal{D}_y + \varphi(b_1(y))\big)\Big) \\ &= \bigcup_{y \in T^{-1}\{x\}} \Big(\mathcal{D}_y + \Xi\big(\bullet b_1(y) b(x)\big)\Big) = \bigcup_{y \in T^{-1}\{x\}} \mathcal{T}_y. \end{aligned}$$

This union is disjoint up to sets of measure zero, by Lemma 5.3.10. A family of sets with this property is called *self-replicating*. Hence, $\mathcal{T}$ is self-replicating.

As a next step in proving that the family $\mathcal{T}$ is a multiple tiling, we will show that $\mathcal{T}$ is quasi-periodic. What we mean by this is the following. Let $\mathbf{y} \in H$ and $r > 0$. We call the set

$$P(B(\mathbf{y}, r)) = \{\mathcal{T}_x \in \mathcal{T} \mid \mathcal{T}_x \cap B(\mathbf{y}, r) \neq \emptyset\}$$

the *local arrangement* in $B(\mathbf{y}, r)$. The family $\mathcal{T}$ is called *quasi-periodic* if for any $r > 0$, there is an $R > 0$ such that for any $\mathbf{y}, \mathbf{y}' \in H$ the local arrangement of $B(\mathbf{y}, r)$ appears up to translation in the ball $B(\mathbf{y}', R)$.

Lemma 5.4.6. *The family $\mathcal{T}$ is quasi-periodic.*

Proof. Since a ball in H can contain only finitely many points from $\Xi(\mathbb{Z}[\beta] \cap X)$ and since there are only finitely many different sets $\mathcal{D}_x$, $x \in \mathbb{Z}[\beta] \cap X$, then by the boundedness of the tiles, for each $r > 0$ there are only finitely many different local arrangements, up to translation.

Let $r > 0$. If two tiles $\mathcal{T}_x = \Xi(x) + \mathcal{D}_x$ and $\mathcal{T}_{x+y} = \Xi(x) + \Xi(y) + \mathcal{D}_{x+y}$ are in the same local arrangement, then

$$\|\Xi(y)\| < 2(r + \max_{w \in {}^\omega I} \|\varphi(w)\|) \quad \text{and} \quad y \in X - X. \tag{5.13}$$

By Lemma 5.4.3 and Lemma 5.4.2 the set of elements $y \in \mathbb{Z}[\beta]$ that satisfy (5.13) is finite. Call this set F. Note that $0 \in F$.

Now, take some local arrangement $P(B(\mathbf{y}, r))$ and $x \in \mathbb{Z}[\beta] \cap X$, $\mathcal{T}_x \in P(B(\mathbf{y}, r))$. Let $y \in F$. If $x + y \in X_{i,n}$ for some $i \in I$, $0 \leq n \leq N_i$, then there is an $\varepsilon_y > 0$, such that $[x + y, x + y + \varepsilon_y) \subseteq X_{i,n}$. If, on the other hand, $x + y \notin X$, then there is an $\varepsilon_y > 0$, such that $[x + y, x + y + \varepsilon_y) \cap X = \emptyset$. Set $\varepsilon = \min_{y \in F} \varepsilon_y$. Then $(x + F + z) \cap X_{i,n} = ((x + F) \cap X_{i,n}) + z$, for all $z \in [0, \varepsilon)$, $i \in I$ and $0 \leq n \leq N_i$. Moreover, if $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$ then $\mathcal{D}_{x+y+z} = \mathcal{D}_{x+y}$ and thus, that $\mathcal{T}_{x+y+z} = \mathcal{T}_{x+y} + \Xi(z) \in P(B(\mathbf{y} + \Xi(z), r))$ for all $y \in F$ with $\mathcal{T}_{x+y} \in P(B(\mathbf{y}, r))$. Also, we can not have any other tile $\mathcal{T}_{x'} \in P(B(\mathbf{y} + \Xi(z), r))$, since this would imply that $x' \in (x+F+z) \cap X$ and thus that $\mathcal{T}_{x'-z} \in P(B(\mathbf{y}, r))$. So, $P(B(\mathbf{y} + \Xi(z), r))$ is up to translation equal to $P(B(\mathbf{y}, r))$.

By Lemma 5.4.3, the set $\Xi\big(\mathbb{Z}[\beta] \cap [0, \varepsilon)\big)$ is relatively dense in H. Thus, every local arrangement occurs relatively densely in H. Since the number of local arrangements is finite, the lemma is proved. □

Lemma 5.4.4 implies that every $\mathbf{y} \in H$ lies in at least one tile. By the locally finiteness of $\mathcal{T}$, there exists an $\textsc{m} \geq 1$ such that all elements of H are contained in at least $\textsc{m}$ tiles and there exist elements of H that are not contained in $\textsc{m}+1$ tiles. We have the following definition.

Definition 5.4.1. A point $\mathbf{y} \in H$ is called an $\textsc{m}$-*exclusive point*, $\textsc{m} \geq 1$, if there are $x_1, \ldots, x_\textsc{m} \in \mathbb{Z}[\beta] \cap X$ with $\mathbf{y} \in \mathcal{T}_{x_k}$ for all $1 \leq k \leq \textsc{m}$ and $\mathbf{y} \notin \mathcal{T}_x$ for all $x \notin \{x_1, \ldots x_\textsc{m}\}$, $x \in \mathbb{Z}[\beta] \cap X$, and if there is no other point $\mathbf{y}' \in H$ that lies in less than $\textsc{m}$ tiles. A point $\mathbf{y} \in H$ is called an *exclusive point* if it is 1-exclusive.

Similarly to Ito and Rao ([IR06]), we obtain the following lemma, which gives (mt4).

Lemma 5.4.7. *There exists an* $\textsc{m} \geq 1$, *such that almost every* $\mathbf{y} \in H$ *is contained in exactly* $\textsc{m}$ *tiles.*

Proof. We first show that the set of points that do not lie on the boundary of a tile is open and of full measure. Let $C = \bigcup_{x \in \mathbb{Z}[\beta] \cap X} \partial \mathcal{T}_x$ denote the union of the boundaries of all the tiles in $\mathcal{T}$. This set is closed, since it is the countable union of closed sets that are locally finite. Hence, $H \backslash C$ is open. By Proposition 5.3.2 we also have $\lambda^{d-1}(C) = 0$.

Suppose that an $\textsc{m}$-exclusive point exists. Let $\bar{\mathbf{y}} \in H$ be an $\textsc{m}$-exclusive point of the tiles $\mathcal{T}_{x_1}, \ldots, \mathcal{T}_{x_\textsc{m}}$. Since the tiles are closed, there is an $\varepsilon > 0$ such that $B(\bar{\mathbf{y}}, \varepsilon) \cap \mathcal{T}_x = \emptyset$ for all $x \in (\mathbb{Z}[\beta] \cap X) \cap \{x_1, \ldots, x_\textsc{m}\}$. We must have $\bar{\mathbf{y}} \in H \backslash C$, for if not, then for any $0 < \varepsilon' < \varepsilon$ we can find a point $\mathbf{y}' \in B(\bar{\mathbf{y}}, \varepsilon')$ that lies in $\textsc{m} - 1$ tiles, contradicting the fact that $\bar{\mathbf{y}}$ is an $\textsc{m}$-exclusive point. Since $H \backslash C$ is open, there is an $\varepsilon^* > 0$, such that $B(\bar{\mathbf{y}}, \varepsilon^*)$ is contained in exactly the tiles $\mathcal{T}_{x_1}, \ldots, \mathcal{T}_{x_\textsc{m}}$. By the self-replicating property, this implies that for each tile $\mathcal{T}_{x_k}$, $M_\beta^{-1}B(\bar{\mathbf{y}}, \epsilon) \subseteq M_\beta^{-1}\mathcal{T}_{x_k}$ and that this set subdivides into tiles from $\mathcal{T}$ with disjoint interior. Hence, almost every point in $M_\beta^{-1}B(\bar{\mathbf{y}}, \varepsilon)$ is also contained in exactly $\textsc{m}$ tiles. The same holds for almost all points from $M_\beta^{-n}B(\bar{\mathbf{y}}, \varepsilon)$ for all $n \geq 1$.

Now take a point $\mathbf{y} \in H \backslash C$. Then, there is an $r > 0$, such that $B(\mathbf{y}, r) \subseteq H \backslash C$. Suppose that $B(\mathbf{y}, r)$ is contained in m tiles. Then $m \geq \textsc{m}$. By the quasi-periodicity, there is an R such that for all $\mathbf{y}' \in H$ there is a $z \in \mathbb{Z}[\beta]$, such that

$$P(B(\mathbf{y}, r)) + \Xi(z) \subseteq P(B(\mathbf{y}', R)).$$

Since the matrix M_β^{-1} is expanding on H, there is an $n \geq 1$, such that $M_\beta^{-n}B(\bar{\mathbf{y}}, \varepsilon)$ contains a ball of radius R. This ball contains a translation of $P(B(\mathbf{y}, r))$, hence $m = \textsc{m}$. □

We have now established all the properties of a multiple tiling.

Theorem 5.4.1. *Let T be a right-continuous β-transformation as in Definition 5.2.3, with β a Pisot unit, I a finite set, $A \subset \mathbb{Q}(\beta)$ and $TX = X$. Suppose that $\mathcal{V}$, as given in (5.10), is a finite set and that the invariant measure $\lambda^d \circ \pi^{-1}$, given by the natural extension, has support X. Then, the family $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a multiple tiling of H.*

Proof. Since $\mathcal{V}$ is finite, the set of prototiles $\{\mathcal{D}_{i,n} \,|\, i \in I, \, 0 \leq n \leq N_i\}$ is finite as well. The proof is then given by Lemma 5.4.5, Corollary 5.4.1, Lemma 5.4.4 and Lemma 5.4.7. □

We have the following immediate corollary.

Corollary 5.4.2. *Let T satisfy the conditions of Theorem 5.4.1. Then the family $\mathcal{T}$ forms a tiling of H if and only if there is a point in H which lies in exactly one tile.*

We would like a way to find $\textsc{m}$-exclusive points. For that we need some results about the points with a periodic T-expansion.

5.4.2 *Periodic expansions*

We first characterize eventually periodic expansions for β-transformations that are right-continuous, as defined in Definition 5.2.3, with β a Pisot number (not necessarily a Pisot unit) and $A \subset \mathbb{Q}(\beta)$. This result was proved e.g. in Frank and Robinson ([FR08]) for a slightly smaller class of transformations, and generalizes Theorem 5.1.1 by Schmidt.

Theorem 5.4.2. *Let T be a right-continuous β-transformation as in Definition 5.2.3, β a Pisot number and $A \subset \mathbb{Q}(\beta)$. Then for $x \in X$, $b(x)$ is eventually periodic if and only if $x \in \mathbb{Q}(\beta)$.*

Proof. If $b(x) = b_1 \cdots b_n (b_{n+1} \cdots b_{n+p})^\omega$ is eventually periodic, then

$$x = \frac{a_{b_1}}{\beta} + \cdots + \frac{a_{b_n}}{\beta^n} + \frac{1}{\beta^p - 1}\left[\frac{a_{b_{n+1}}}{\beta^{n-p+1}} + \frac{a_{b_{n+2}}}{\beta^{n-p+2}} + \cdots + \frac{a_{b_{n+p}}}{\beta^n}\right],$$

which is clearly in $\mathbb{Q}(\beta)$.

For the other implication, let $x \in \mathbb{Q}(\beta) \cap X$. Since I is a finite set, there is a $q \in \mathbb{Z}\backslash\{0\}$, such that $x, a_i \in q^{-1}\mathbb{Z}[\beta]$ for all $i \in I$. For each $k \geq 0$, $T^k x \in q^{-1}\mathbb{Z}[\beta] \cap X$. Furthermore,

$$\|\Xi(T^k x)\| = \|M_\beta^k \Xi(x) - \varphi(b_1(x) \cdots b_k(x))\| \leq \|\Xi(x)\| + \max_{w \in {}^\omega I} \|\varphi(w)\|.$$

Then, by Lemma 5.4.3 and the injectivity of Ξ, the set $\{T^k x \,|\, k \geq 0\}$ is finite, which implies that $b(x)$ is eventually periodic. □

Now we turn to purely periodic expansions. Here, T satisfies the assumptions of Section 5.3, i.e., β is a Pisot unit and $A \subset \mathbb{Q}(\beta)$. Denote by $\mathcal{P}$ the set of points of which the expansion generated by T is purely periodic, i.e., $x \in \mathcal{P}$ if and only if there exists some $n \geq 1$ such that $T^n x = x$. Clearly, $\mathcal{P} \subset \mathbb{Q}(\beta)$. We will consider also the sets $\mathcal{P}_q = \mathcal{P} \cap q^{-1}\mathbb{Z}[\beta]$, $q \in \mathbb{Z}_{>0}$.

Lemma 5.4.8. *The origin* $\mathbf{0}$ *belongs to* $\mathcal{T}_x$, $x \in \mathbb{Q}(\beta) \cap X$, *if and only if* $x \in \mathcal{P}$.

Proof. Suppose that $x \in \mathcal{P}$, i.e., $b(x) = (b_1 \cdots b_n)^\omega$ for some $n \geq 1$. Then we clearly have $x \in \mathbb{Q}(\beta) \cap X$ and ${}^\omega(b_1 \cdots b_n) \cdot (b_1 \cdots b_n)^\omega \in \mathcal{S}$. Furthermore,

$$\begin{aligned}\varphi\Big({}^\omega\big(b_1 \cdots b_n\big)\Big) &= \sum_{j=2}^{d} \big(\Gamma_j(a_{b_1})\beta_j^{n-1} + \cdots + \Gamma_j(a_{b_n})\big)(1 + \beta_j^n + \beta_j^{2n} + \cdots)\mathbf{v}_j \\ &= \sum_{j=2}^{d} \Gamma_j\left(\frac{a_{b_1}\beta^{n-1} + \cdots + a_{b_n}}{1 - \beta^n}\right)\mathbf{v}_j = -\sum_{j=2}^{d} \Gamma_j(x)\mathbf{v}_j = -\Xi(x).\end{aligned}$$

Hence, $\mathbf{0} = \Xi(x) + \varphi\big({}^\omega(b_1 \cdots b_n)\big)$ and thus $\mathbf{0} \in \mathcal{T}_x$.

Now, take an $x \in \mathbb{Q}(\beta) \cap X$ with $\mathbf{0} \in \mathcal{T}_x$. Let $q \in \mathbb{Z}$ be such that for all $i \in I$, $x, a_i \in q^{-1}\mathbb{Z}[\beta] \cap X$. Then there exists some $w \in {}^\omega I$ such that $w \cdot b(x) \in \mathcal{S}$ and $\varphi(w) + \Xi(x) = \mathbf{0}$. For $k \geq 0$, let $x_k = .w_{-k} \cdots w_0 b(x) \in q^{-1}\mathbb{Z}[\beta] \cap X$ and $w^{(k)} = \cdots w_{-k-2}w_{-k-1}$. Then we have

$$\varphi(w^{(k)}) + \Xi(x_k) = M_\beta^{-k-1}(\varphi(w) + \Xi(x)) = \mathbf{0},$$

thus $\mathbf{0} \in \mathcal{T}_{x_k}$ for all $k \geq 0$. Since $x_k \in q^{-1}\mathbb{Z}[\beta] \cap X$ and the set $\mathcal{D}_{x_k}$ is bounded for each $k \geq 0$, we obtain by Lemma 5.4.3 that $\{\Xi(x_k) \,|\, k \geq 0\}$ is a finite set. By the injectivity of Ξ, then also the set $\{x_k \,|\, k \geq 0\}$ is finite and hence $x_m = x_n$ for some $m > n \geq 0$. Since $T^{m-n}x_m = x_n$ and $T^m x_m = x$, we obtain $x \in \mathcal{P}$. □

Since $\Xi(q^{-1}\mathbb{Z}[\beta] \cap X)$ is uniformly discrete, Lemma 5.4.8 implies that $\mathcal{P}_q$ is finite for each $q \geq 1$. The next proposition gives a simple characterization of all the purely periodic points, cf. [IR06] by Ito and Rao.

Theorem 5.4.3. *We have* $x \in \mathcal{P}$ *if and only if* $x \in \mathbb{Q}(\beta)$ *and* $x\mathbf{v}_1 + \Xi(x) \in \widehat{X}$.

Proof. If $x \in \mathcal{P}$, then we clearly have $x \in \mathbb{Q}(\beta) \cap X$. By Lemma 5.4.8, $x \in \mathcal{P}$ is equivalent with $\mathbf{0} \in \mathcal{T}_x$, but this means that $-\Xi(x) \in \mathcal{D}_x$, i.e., $x\mathbf{v}_1 + \Xi(x) \in \widehat{X}$ by (5.9). □

Note that this theorem gives a nice characterization of rational numbers with purely periodic β-expansions generated by T, since we have $\Gamma_j(x) = x$ for $x \in \mathbb{Q}$ and thus $x\mathbf{v}_1 + \Xi(x) = x\mathbf{v}_1 + \sum_{j=2}^{d} x\mathbf{v}_j = (x, 0, \ldots, 0)$.

5.4.3 *To find an* ᴍ*-exclusive point*

To determine the covering degree of the multiple tiling, we need to find a point in H that is an ᴍ-exclusive point for some ᴍ ≥ 1. In this section we let T be a right-continuous β-transformation with β a Pisot unit and, for convenience, $A \subset \mathbb{Z}[\beta]$.

Recall that $\mathcal{P}_1$ is the set of points in $\mathbb{Z}[\beta]$ of which the expansion generated by T is purely periodic. Note that $\mathbf{0}$ is an exclusive point if and only if $\mathcal{P}_1$ consists only of one element. This generalizes the (F) property, see (5.1). If $\mathbf{0}$ is contained in more than one tile, it is more difficult to determine the covering degree ᴍ. Corollary 5.4.3 provides an easy way to determine the number of tiles to which a point belongs. We restrict to points $\Xi(z)$, $z \in \mathbb{Z}[\beta] \cap [0, \infty)$, because of the following lemma. Compare Theorem 5.1.3.

Lemma 5.4.9 (Akiyama, [Aki99]). *The set $\Xi\big(\mathbb{Z}[\beta] \cap [0, \infty)\big)$ is dense in H.*

Proof. By Lemma 5.4.3, $\Xi\big(\mathbb{Z}[\beta] \cap [0, 1)\big)$ is relatively dense in H. We have

$$\Xi\big(\mathbb{Z}[\beta] \cap [0, \infty)\big) = \bigcup_{k \geq 0} \Xi\big(\mathbb{Z}[\beta] \cap [0, \beta^k)\big) = \bigcup_{k \geq 0} M_\beta^k \Xi\big(\mathbb{Z}[\beta] \cap [0, 1)\big).$$

Since M_β is contracting, we obtain that $\Xi\big(\mathbb{Z}[\beta] \cap [0, \infty)\big)$ is dense in H. □

Proposition 5.4.1. *We have $\Xi(z) \in \mathcal{T}_x$ for $z \in \mathbb{Z}[\beta] \cap [0, \infty)$, $x \in \mathbb{Z}[\beta] \cap X$, if and only if there exists some $p \in \mathcal{P}_1$ and some $\kappa \geq 0$ such that*

$$T^k(p_k + \beta^{-k} z) = x \quad \textit{for all } k \geq \kappa, \tag{5.14}$$

where $p_k \in \mathcal{P}_1$ is defined by $T^k p_k = p$.

Proof. Let $\Xi(z) \in \mathcal{T}_x$, which means that $\Xi(z) = \Xi(x) + \varphi(w)$ for some $w \in {}^\omega I$ with $w \cdot b(x) \in \mathcal{S}$, and set $x_k = .w_{-k+1} \cdots w_0 b(x)$ for $k \geq 1$ and $x_0 = x$. Then

$$\Xi(x_k - \beta^{-k} z) = \Xi(x_k) - M_\beta^{-k}\big(\Xi(x) + \varphi(w)\big) = -\varphi(\cdots w_{-k-1} w_{-k}). \tag{5.15}$$

Hence, for each k, $\|\Xi(x_k - \beta^{-k} z)\| \leq \max_{w \in {}^\omega I} \|\varphi(w)\|$ and thus by Lemma 5.4.3 the set $\{x_k - \beta^{-k} z \,|\, k \geq 0\}$ is finite. This means that there exists some y such that $x_k - \beta^{-k} z = y$ for infinitely many $k \geq 1$. Since $x_k \in X$ and the intervals X_i, $i \in I$, are left-closed, we obtain $y \in X$.

Let $K = \{k \geq 0 \,|\, x_k - \beta^{-k} z = y\}$. Since the intervals $X_{i,n}$ are right open, there is a κ_1, such that $\mathcal{D}_y = \mathcal{D}_{y+\beta^{-k} z}$ for all $k \geq \kappa_1$. Then $\mathcal{D}_{x_k - \beta^{-k} z} = \mathcal{D}_{x_k}$ for all $k \in K$, $k \geq \kappa_1$. Since $\varphi(\cdots w_{-k-1} w_{-k}) \in \mathcal{D}_{x_k}$, we obtain $\mathbf{0} \in \mathcal{T}_{x_k - \beta^{-k} z}$ by (5.15). By Lemma 5.4.8, then for each $k \in K$, $k \geq \kappa_1$, there exists some $p_k \in \mathcal{P}_1$ such that $x_k - \beta^{-k} z = p_k$, which implies $T^k(p_k + \beta^{-k} z) = T^k x_k = x$.

By the finiteness of $\mathcal{P}_1$, there is a $\kappa_2 \geq 0$, such that for each $k \geq \kappa_2$, $p' \in \mathcal{P}_1$,

$$T(p' + \beta^{-k} z) = Tp' + \beta^{-k+1} z. \tag{5.16}$$

Let κ be the smallest $k \in K$ with $k \geq \max\{\kappa_1, \kappa_2\}$. Let $p = T^\kappa p_\kappa$ and for each $k \geq 0$, let p_k be the unique element from $\mathcal{P}_1$, such that $T^k p_k = p$. Then, by (5.16), for each $k \geq \kappa$ we have

$$T^k(p_k + \beta^{-k} z) = T^\kappa(p_\kappa + \beta^{-\kappa} z) = T^\kappa x_\kappa = x,$$

which gives (5.14).

For the other direction, assume that (5.14) holds. Then we have sequences $w^{(k)} \in {}^\omega I$, $k \geq \kappa$, such that $w^{(k)} \cdot b(p_k + \beta^{-k} z) \in \mathcal{S}$. Set

$$\mathbf{y}_k = M_\beta^k \big(\varphi(w^{(k)}) + \Xi(p_k + \beta^{-k} z)\big) = \Xi(z) + M_\beta^k \big(\varphi(w^{(k)}) + \Xi(p_k)\big).$$

Then, $\mathbf{y}_k \in \mathcal{T}_{T^k(p_k + \beta^{-k} z)} = \mathcal{T}_x$ and $\lim_{k\to\infty} \mathbf{y}_k = \Xi(z)$. Thus $\Xi(z) \in \mathcal{T}_x$. □

Corollary 5.4.3. *Let $z \in \mathbb{Z}[\beta] \cap [0, \infty)$ and $\kappa \geq 0$ such that $p + \beta^{-\kappa} z \in X$ and $b_1(p + \beta^{-\kappa-1} z) = b_1(p)$ for all $p \in \mathcal{P}_1$. Then $\Xi(z)$ lies exactly in the tiles $\mathcal{T}_{T^\kappa(p+\beta^{-\kappa} z)}$, $p \in \mathcal{P}_1$.*

Proof. Note that $b_1(p + \beta^{-\kappa-1} z) = b_1(p)$ implies $b_1(p + \beta^{-k} z) = b_1(p)$ for all $k > \kappa$, and thus $T^{k-\kappa}(p + \beta^{-k} z) = T^{k-\kappa} p + \beta^{-\kappa} z$ for all $p \in \mathcal{P}_1$. If we define $p_k \in \mathcal{P}_1$ by $T^k p_k = p'$ for some fixed $p' \in \mathcal{P}_1$, then we have therefore for all $k \geq \kappa$,

$$T^k(p_k + \beta^{-k} z) = T^\kappa(T^{k-\kappa} p_k + \beta^{-\kappa} z) = T^\kappa(p_\kappa + \beta^{-\kappa} z).$$

Hence, by Proposition 5.4.1, $\Xi(z) \in \mathcal{T}_{T^\kappa(p_\kappa + \beta^{-\kappa} z)}$. For every $p \in \mathcal{P}_1$ we have $T^\kappa p = p'$ for some $p' \in \mathcal{P}_1$. Hence, $\Xi(z) \in \mathcal{T}_{T^\kappa(p+\beta^{-\kappa} z)}$ for all $p \in \mathcal{P}_1$. By Proposition 5.4.1, $\Xi(z)$ cannot lie in other tiles. □

The main difficulty is to find some $z \in \mathbb{Z}[\beta] \cap [0, \infty)$ such that the number of tiles to which $\Xi(z)$ belongs is equal to the covering degree, i.e., such that $\Xi(z)$ does not lie on the boundary of two tiles. By Corollary 5.4.3 it is relatively easy to find a point lying in exactly M tiles. It is usually harder to show that M is the covering degree of the multiple tiling, i.e., that there exists an open ball lying in exactly M tiles. In case M = 1, however, the proof of the following proposition shows how to construct such a point from points with weaker properties. This generalizes the (W) property, see (5.2), and Theorem 5.1.6 where $p_0 = 0$, since $T^k(p + z) = 0$ implies that the classical greedy β-expansion of $p + z$ is finite.

Proposition 5.4.2. *Let $\varepsilon = \min_{p \in \mathcal{P}_1} (r_{b_1(p)} - p)\beta$. Then $\mathcal{T}_{p_0}$, $p_0 \in \mathcal{P}_1$, contains an exclusive point if and only if for every $p \in \mathcal{P}_1$ there exists some $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$ and some $k \geq 0$ such that*

$$T^k(p + z) = T^k(p_0 + z) = p_0. \tag{5.17}$$

Proof. If $\mathcal{T}_{p_0}$ contains an exclusive point, then it contains an exclusive point $\Xi(z')$, $z' \in \mathbb{Z}[\beta] \cap [0, \infty)$ by Lemma 5.4.9. For z', there is a $\kappa > 0$, such that the conditions of Corollary 5.4.3 are satisfied. Then, by Corollary 5.4.3, we have that $T^\kappa(p + \beta^{-\kappa} z') = p_0$ for all $p \in \mathcal{P}$, and $b_1(p + \beta^{-\kappa-1} z') = b_1(p)$. This implies $\beta^{-\kappa-1} z' < r_{b_1(p)} - p$, hence we can choose $z = \beta^{-\kappa} z'$ and $k = \kappa$.

For the converse, let $\mathcal{P}_1 = \{p_0, p_1, \ldots, p_h\}$ and assume that for every $p \in \mathcal{P}_1$ there exist a $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$ and a $k \geq 0$, such that (5.17) holds. Note that for

$h = 0$, $\mathbf{0}$ is an exclusive point of $\mathcal{T}_{p_0}$. For $h \geq 1$, we will recursively construct a point y_h, that satisfies the conditions of Corollary 5.4.3 for some κ and for which $T^\kappa(p + \beta^{-\kappa} y_h) = p_0$ for all $p \in \mathcal{P}_1$. Then Corollary 5.4.3 gives that this point y_h is an exclusive point of the tile $\mathcal{T}_{p_0}$.

First note that if (5.17) holds for some p, k, z, then it also holds for $z' = \beta^{-n} z$ and $k' = k + n$ if n is a multiple of the period lengths of p and p_0, since $T^n(p + \beta^{-n} z) = T^n p + z = p + z$ and $T^n(p_0 + \beta^{-n} z) = p_0 + z$. Therefore, we can find arbitrarily small z such that (5.17) holds. For every such z, we can find arbitrarily large k such that (5.17) holds, since $b(p_0)$ is purely periodic.

For $n = 1$, choose $k_1 \geq 1$ and $z_1 \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$, such that

- $T^{k_1}(p_1 + z_1) = T^{k_1}(p_0 + z_1) = p_0$, which can be done by (5.17).
- $b_1(p + \beta^{-1} z_1) = b_1(p)$ for all $p \in \mathcal{P}_1$, which can be done by choosing z_1 sufficiently small.
- $T^{k_1}(p_2 + z_1) \in \mathcal{P}_1$, which can be done by choosing k_1 sufficiently large, since by Theorem 5.4.2 the expansion of $p_2 + z_1$ generated by T is eventually periodic.

Now, suppose that for $1 \leq n < h$ we have $k_n \geq 0$ and $z_n \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$ that satisfy all of the following, where $y_0 = 0$, $y_n = \beta^{k_n}(y_{n-1} + z_n)$ and $s_n = k_1 + \cdots + k_n$.

(i) $T^{k_n}(T^{s_{n-1}}(p_n + \beta^{-s_{n-1}} y_{n-1}) + z_n) = T^{k_n}(p_0 + z_n) = p_0$,

(ii) $b_1(p + \beta^{-s_n - 1} y_n) = b_1(p)$ for all $p \in \mathcal{P}_1$,

(iii) $T^{s_{n-1}}(p + \beta^{-s_n} y_n) = T^{s_{n-1}}(p + \beta^{-s_{n-1}} y_{n-1}) + z_n$ for all $p \in \mathcal{P}_1$. In particular, this means that we assume that $p + \beta^{-s_n} y_n \in X$.

(iv) $T^{s_n}(p_{n+1} + \beta^{-s_n} y_n) \in \mathcal{P}_1$.

The previously chosen k_1 and z_1 satisfy these properties. Then, for $n + 1$, by (iv) and (5.17), we have arbitrarily large $k_{n+1} \geq 0$ and arbitrarily small $z_{n+1} \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$, such that

$$T^{k_{n+1}}(T^{s_n}(p_{n+1} + \beta^{-s_n} y_n) + z_{n+1}) = T^{k_{n+1}}(p_0 + z_{n+1}) = p_0.$$

If we set $y_{n+1} = \beta^{k_{n+1}}(y_n + z_{n+1})$ and $s_{n+1} = s_n + k_{n+1}$, then by choosing z_{n+1} small enough, and by (ii), we can get for all $p \in \mathcal{P}_1$, that

$$b_1(p + \beta^{-s_{n+1} - 1} y_{n+1}) = b_1(p + \beta^{-s_n - 1} y_n + \beta^{-s_n - 1} z_{n+1}) = b_1(p).$$

By choosing z_{n+1} small enough, we also have

$$T^{s_n}(p + \beta^{-s_{n+1}} y_{n+1}) = T^{s_n}(p + \beta^{-s_n} y_n + \beta^{-s_n} z_{n+1}) = T^{-s_n}(p + \beta^{-s_n} y_n) + z_{n+1}$$

for all $p \in \mathcal{P}_1$. If $n + 1 < h$, by Theorem 5.4.2 we can choose k_{n+1} large enough, such that

$$T^{s_{n+1}}(p_{n+2} + \beta^{-s_{n+1}} y_{n+1}) \in \mathcal{P}_1.$$

So, for $1 \leq n < h$, k_n and z_n that satisfy (i)-(iv) exist and we have k_h and z_h that satisfy (i)-(iii).

Note that by (ii) and (iii), y_h satisfies all the conditions of Corollary 5.4.3 with $\kappa = s_h$. Furthermore, by (iii) and (i), we have for $1 \le n \le h$,

$$\begin{aligned} T^{s_h}(p_n + \beta^{-s_h} y_h) &= T^{k_h}\big(T^{s_{h-1}}(p_n + \beta^{-s_{h-1}} y_{h-1}) + z_h\big) \\ &= T^{k_h}\Big(\cdots\Big(T^{k_{n+1}}\Big(T^{k_n}\Big(T^{s_{n-1}}(p_n + \beta^{-s_{n-1}} y_{n-1}) + z_n\Big) + z_{n+1}\Big)\cdots\Big) + z_h\Big) \\ &= T^{k_h}(\cdots(T^{k_{n+1}}(p_0 + z_{n+1})\cdots) + z_h) = p_0. \end{aligned}$$

Similarly, we obtain by (5.17) that

$$T^{s_h}(p_0 + \beta^{-s_h} y_h) = T^{k_h}(\cdots(T^{k_2}(T^{k_1}(p_0 + z_1) + z_2)\cdots) + z_h) = p_0.$$

Hence, by Corollary 5.4.3, y_h is an exclusive point of $\mathcal{T}_{p_0}$. $\square$

Remark 5.4.1. The condition given by Proposition 5.4.2 corresponds to the (W′) property, see (5.3). Note that it is required in (W′) that there exists $z \in \mathbb{Z}[\beta] \cap [0, \varepsilon)$ satisfying (5.17) for every $\varepsilon > 0$. As Proposition 5.4.2 shows, this requirement is not necessary, and $\varepsilon = \min_{p \in \mathcal{P}_1}(r_{b_1(p)} - p)\beta$ suffices.

5.4.4 *Tiling of the torus*

Assume that $T : X \to X$ satisfies all the conditions from Theorem 5.4.1, but with $A \subset \mathbb{Z}[\beta]$. Similarly to Ito and Rao ([IR06]), we can show that $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a tiling of H if and only if $\widehat{Y} + \mathbb{Z}^d$ is a tiling of $\mathbb{R}^d$, where $\widehat{Y}$ is the closure of the natural extension domain $\widehat{X}$ as defined in Section 5.3. By Lemma 5.3.5, we know that $\mathbb{R}^d$ is covered by the set $\widehat{Y} + \mathbb{Z}^d$.

Proposition 5.4.3 (cf. Ito and Rao ([IR06]))**.** *The family $\{\mathbf{z} + \widehat{Y}\}_{\mathbf{z} \in \mathbb{Z}^d}$ forms a tiling of $\mathbb{R}^d$ if and only if the family $\mathcal{T} = \{\mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ forms a tiling of H.*

Proof. Assume that $\{\mathbf{z} + \widehat{Y}\}_{\mathbf{z} \in \mathbb{Z}^d}$ forms a tiling of $\mathbb{R}^d$. Since $\widehat{X}$ differs from $\widehat{Y}$ only by a set of measure zero, we can consider $\widehat{X}$ instead of $\widehat{Y}$.

Let $\mathbf{z} \in \mathbb{Z}^d$, i.e., $\mathbf{z} = -x\mathbf{v}_1 - \Xi(x)$, $x \in \mathbb{Z}[\beta]$. If $(\mathbf{z} + \widehat{X}) \cap H \neq \emptyset$, then there is an $x' \in X$, such that $-x + x' = 0$, hence $x \in X$. Moreover,

$$(\mathbf{z} + \widehat{X}) \cap H = -\Xi(x) - \mathcal{D}_x = -\mathcal{T}_x.$$

Suppose that $\mathcal{T}$ is not a tiling of H. Then $\lambda^{d-1}(\mathcal{T}_x \cap \mathcal{T}_{x'}) > 0$ for some $x, x' \in \mathbb{Z}[\beta] \cap X$, $x' \neq x$. Let $x \in X_{i,n}$, $x' \in X_{i',n'}$, and set $\mathbf{z}' = -x'\mathbf{v}_1 - \Xi(x')$. Then $[x, r_{i,n})\mathbf{v}_1 - \mathcal{D}_x \subseteq \widehat{X}$ and

$$[x, r_{i,n})\mathbf{v}_1 - \mathcal{D}_x + \mathbf{z} = [0, r_{i,n} - x)\mathbf{v}_1 - \mathcal{T}_x \subseteq \mathbf{z} + \widehat{X}.$$

Similar statements hold for x' and $\mathbf{z}'$. From $\lambda^{d-1}(\mathcal{T}_x \cap \mathcal{T}_{x'}) > 0$, we see that

$$\lambda^d\big(([0, r_{i,n} - x)\mathbf{v}_1 - \mathcal{T}_x) \cap ([0, r_{i',n'} - x')\mathbf{v}_1 - \mathcal{T}_{x'})\big) > 0.$$

Since

$$\big([0, r_{i,n} - x)\mathbf{v}_1 - \mathcal{T}_x\big) \cap \big([0, r_{i',n'} - x')\mathbf{v}_1 - \mathcal{T}_{x'}\big) \subseteq (\mathbf{z} + \widehat{X}) \cap (\mathbf{z}' + \widehat{X}),$$

then also $\lambda^d\big((\mathbf{z} + \widehat{X}) \cap (\mathbf{z}' + \widehat{X})\big) > 0$, contradicting that $\{\mathbf{z} + \widehat{Y}\}_{\mathbf{z} \in \mathbb{Z}^d}$ is a tiling.

Assume now that $\mathcal{T}$ forms a tiling of H. Consider the hyperplane $y\mathbf{v}_1 + H$ for some $y \in \mathbb{Z}[\beta]$. Let $\mathbf{z} = -z\mathbf{v}_1 - \Xi(z)$, $z \in \mathbb{Z}[\beta]$ be such that $(\mathbf{z} + \widehat{X}) \cap (y\mathbf{v}_1 + H) \neq \emptyset$. Then we have $z + y \in X$ and

$$(\mathbf{z} + \widehat{X}) \cap (y\mathbf{v}_1 + H) = y\mathbf{v}_1 - \Xi(z) - \mathcal{D}_{z+y} = y\mathbf{v}_1 + \Xi(y) - \mathcal{T}_{z+y}.$$

Since $\{y\mathbf{v}_1 + \Xi(y) - \mathcal{T}_x\}_{x \in \mathbb{Z}[\beta] \cap X}$ is a tiling of $y\mathbf{v}_1 + H$, we have that

$$\lambda^d((\mathbf{z} + \widehat{X}) \cap (\mathbf{z}' + \widehat{X}) \cap (y\mathbf{v}_1 + H)) = 0$$

for all $\mathbf{z}, \mathbf{z}' \in \mathbb{Z}^d$ with $\mathbf{z} \neq \mathbf{z}'$ and $y \in \mathbb{Z}[\beta]$. Since $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$, and $\widehat{X}$ has the shape given in (5.9), we obtain that $\lambda^d\big((\mathbf{z} + \widehat{X}) \cap (\mathbf{z}' + \widehat{X})\big) = 0$, which implies $\lambda^d\big((\mathbf{z} + \widehat{Y}) \cap (\mathbf{z}' + \widehat{Y})\big) = 0$. All other tiling conditions for $\{\mathbf{z} + \widehat{Y}\}_{\mathbf{z} \in \mathbb{Z}^d}$ follow from the compactness of $\widehat{Y}$, Lemma 5.3.5 and Lemma 5.4.5. □

The next lemma shows that, when $\{\mathbf{z} + \widehat{Y}\}_{\mathbf{z} \in \mathbb{Z}^d}$ forms a tiling of $\mathbb{R}^d$, the partition $\{\widehat{X}_i\}_{i \in I}$ of $\widehat{X}$ can be used to determine the k-th digit $b_k(x)$ of the expansion of x generated by T up to an error term of order ρ^{k-1} with $\rho = \max_{2 \leq j \leq d} |\beta_j| < 1$.

Lemma 5.4.10. *For every $x \in X$, $k \geq 1$, we have*

$$\beta^{k-1} x \mathbf{v}_1 \in \widehat{X}_{b_k(x)} + M_\beta^{k-1} \mathcal{D}_x \pmod{\mathbb{Z}^d}.$$

If $\mathbf{0} \in \mathcal{D}_x$, then we have $\beta^{k-1} x \mathbf{v}_1 \in \widehat{X}_{b_k(x)} \pmod{\mathbb{Z}^d}$.

Proof. By definition, we have $T^{k-1}x \in X_{b_k(x)}$. Since $TX = X$, there exists some $w \in {}^{\omega}I$ such that $w \cdot b(x) \in \mathcal{S}$ and thus $\varphi(w) \in \mathcal{D}_x$. Since $A \subset \mathbb{Z}[\beta]$, we have $\psi(\sigma u) = \widehat{T}\psi(u) \equiv M_\beta \psi(u) \pmod{\mathbb{Z}^d}$. Then

$$\begin{aligned} \beta^{k-1} x \mathbf{v}_1 - M_\beta^{k-1} \varphi(w) &= M_\beta^{k-1} \psi\big(w \cdot b(x)\big) \\ &\equiv \psi\big(w b_1(x) \cdots b_{k-1}(x) \cdot b(T^{k-1}x)\big) \pmod{\mathbb{Z}^d}, \end{aligned}$$

and $\psi\big(w b_1(x) \cdots b_{k-1}(x) \cdot b(T^{k-1}x)\big) \in \widehat{X}_{b_k(x)}$. Thus, we have the first part of the lemma. If $\mathbf{0} \in \mathcal{D}_x$, then we can choose w, such that $\varphi(w) = \mathbf{0}$. □

In certain cases, we can determine $b_k(x)$ exactly, just by looking at the position of $\beta^{k-1} x \mathbf{v}_1$ modulo $\mathbb{Z}^d$.

Proposition 5.4.4. *Assume that there is a $p_0 \in \mathcal{P}_1$, such that $\mathcal{T}_{p_0}$ contains an exclusive point and that $\mathbf{0} \in \mathcal{D}_x$ for all $x \in X$. Then we have for all $x \in X$, $k \geq 1$, that*

$$b_k(x) = i \quad \textit{if and only if} \quad \beta^{k-1} x \mathbf{v}_1 \in \widehat{X}_i \pmod{\mathbb{Z}^d}.$$

Proof. First notice that the assumptions imply that $\widehat{Y} + \mathbb{Z}^d$ tiles $\mathbb{R}^d$. By the proof of Lemma 5.4.10 we have $\beta^{k-1} x \mathbf{v}_1 \equiv \psi\big({}^{\omega}0 b_1(x) \cdots b_{k-1}(x) \cdot b(T^{k-1}x)\big) \pmod{\mathbb{Z}^d}$. Since $\mathbf{0}$ is an inner point of $\mathcal{D}_x$ and $\varphi\big(b_1(x) \cdots b_k(x)\big) + M_\beta^{k-1} \mathcal{D}_x \subseteq \mathcal{D}_{T^{k-1}x}$, the point $\varphi\big(b_1(x) \cdots b_{k-1}(x)\big)$ is an inner point of $\mathcal{D}_{T^{k-1}x}$. Since every interval X_i is right-open and by Lemma 5.4.5 this implies that $\beta^{k-1} x \mathbf{v}_1$ can lie in $\widehat{X}_{b_k(x)} + \mathbf{z}$ for only one $\mathbf{z} \in \mathbb{Z}^d$. □

5.4.5 *Examples of (multiple) tilings*

We have seen in Section 5.4.4 that it is equivalent to study the tiling property of $\{\mathbf{z} + \widehat{Y}\}_{\mathbf{z}\in\mathbb{Z}^d}$ and of $\mathcal{T}$. We will therefore give pictures of torus (multiple) tilings in case $d = 2$ and of (multiple) tilings of H in case $d = 3$.

In Figure 5.8, we see $\widehat{X}$ and its translates by the vectors $(0,1)$, $(1,0)$ and $(1,1)$ for Example 5.3.1, Example 5.3.3 and Example 5.3.4. Therefore these examples provide tilings. To give a rigorous proof, note that if $\beta = \frac{\sqrt{5}+1}{2}$ and $I = A \subseteq \{-1,0,1\}$, then, since $|\beta_2| = 1/\beta$, we have $\varphi(w) \subseteq \left[\frac{-1}{1-1/\beta}, \frac{1}{1-1/\beta}\right]\mathbf{v}_2$. Thus, for all $p \in \mathcal{P}_1$ we have $|\Gamma_2(p)| \le \frac{1}{1-1/\beta} = \beta^2$. The only points $x \in \mathbb{Z}[\beta]$ with $|x| < 1$ and $|\Gamma_2(x)| \le \beta^2$, are $x = 0$, $x = \pm(\beta - 1) = \pm 1/\beta$, $x = \pm(\beta - 2) = \pm 1/\beta^2$. We immediately obtain that $\mathcal{P}_1 = \{0\}$ in Example 5.3.1 and Example 5.3.3, thus here $\mathbf{0}$ is an exclusive point of $\mathcal{T}_0$.

In Example 5.3.4, we have $\mathcal{P}_1 = \{\pm 1/\beta^2\}$, if we restrain X to the support of the invariant measure, $\left[-\frac{1}{2}, -\frac{1}{2\beta^2}\right) \cup \left[\frac{1}{2\beta^2}, \frac{1}{2}\right)$. Let $z = 1/\beta^2$ and $\kappa = 3$. Then we have

$$T^\kappa(1/\beta^2 + z/\beta^\kappa) = T^3(2/\beta^3) = T^2(-1/\beta^3) = T(-1/\beta^2) = 1/\beta^2,$$
$$T^\kappa(-1/\beta^2 + z/\beta^\kappa) = T^3(-2/\beta^4) = T^2(-2/\beta^3) = T(1/\beta^3) = 1/\beta^2.$$

Since $b_1(1/\beta^2 + 1/\beta^6) = 1 = b_1(1/\beta^2)$ and $b_1(-1/\beta^2 + 1/\beta^6) = \bar{1} = b_1(-1/\beta^2)$, we obtain by Corollary 5.4.3 that $\Xi(1/\beta^2) = \beta^2\mathbf{v}_2$ is an exclusive point of $\mathcal{T}_{1/\beta^2}$.

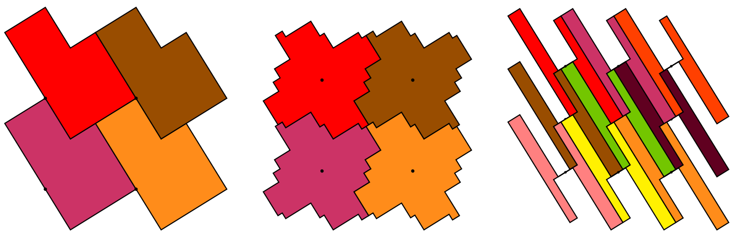


Figure 5.8 The sets $\widehat{X}$ and their translates by the vectors $(1,0)$, $(0,1)$ and $(1,1)$ for Example 5.3.1, Example 5.3.3 and Example 5.3.4.

Figure 5.10 shows that Example 5.3.2 gives a tiling with covering degree 4. Here, we have $\mathcal{P}_1 = \{-1, -1/\beta, -1/\beta^2, 0, 1/\beta^2\}$. It is easy to find a point lying in exactly 4 tiles by using Corollary 5.4.3. Take, for example, $z = 1$ and $\kappa = 4$, then $\Xi(1) \in \mathcal{T}_p$ for $p \in \left\{-1, -\frac{1}{\beta}, 0, \frac{1}{\beta^2}\right\}$. This implies that the covering degree is at most 4. It is slightly harder to show that some open ball is covered 4 times. Remember that $\mathcal{V} = \{-1, -1/\beta, 0, 1/\beta, 1\}$. By (5.12), we have the decompositions

$$\mathcal{D}_{-1} = -\frac{\mathcal{D}_0}{\beta} + \mathbf{v}_2, \qquad \mathcal{D}_{-1/\beta} = \left(-\frac{\mathcal{D}_{-1}}{\beta} - \mathbf{v}_2\right) \cup \left(-\frac{\mathcal{D}_{1/\beta^3}}{\beta} + \mathbf{v}_2\right),$$
$$\mathcal{D}_{1/\beta} = -\frac{\mathcal{D}_{-1/\beta^3}}{\beta} - \mathbf{v}_2, \qquad \mathcal{D}_0 = \left(-\frac{\mathcal{D}_{-1/\beta}}{\beta} - \mathbf{v}_2\right) \cup \left(-\frac{\mathcal{D}_{1/\beta}}{\beta} + \mathbf{v}_2\right).$$

Since $\mathcal{D}_{1/\beta^3} = \mathcal{D}_0$ and $\mathcal{D}_{-1/\beta^3} = \mathcal{D}_{-1/\beta}$, we can write these sets as in (4.2) with the graph from Figure 5.9. Hence, these sets are the unique family of graph-directed sets for a graph-directed IFS, see Definition 4.5.4. We can

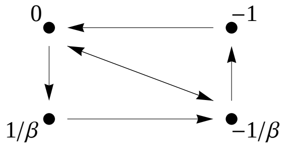


Figure 5.9 The graph for the sets $\mathcal{D}_{-1}$, $\mathcal{D}_{-1/\beta}$, $\mathcal{D}_0$ and $\mathcal{D}_{1/\beta}$.

check that the sets

$$\mathcal{D}_{-1} = [-1/\beta, \beta^2]\mathbf{v}_2,\ \mathcal{D}_{-1/\beta} = \mathcal{D}_0 = [-\beta^2, \beta^2]\mathbf{v}_2,\ \mathcal{D}_{1/\beta} = [-\beta^2, 1/\beta]\mathbf{v}_2,$$

satisfy the set equations above. Therefore $[0, \beta]\mathbf{v}_2$ is covered 4 times by $\mathcal{T}_0 = [-\beta^2, \beta^2]\mathbf{v}_2$, $\mathcal{T}_{-1} = [-\beta, \beta]\mathbf{v}_2$, $\mathcal{T}_{-1/\beta} = [-1, \beta^3]\mathbf{v}_2$, $\mathcal{T}_{1/\beta^2} = [0, 2\beta^2]\mathbf{v}_2$.

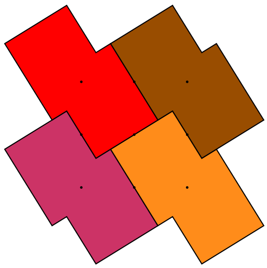


Figure 5.10 The set $\widehat{X}$ and its translates by the vectors in $(2, 0)$, $(0, 2)$ and $(2, 2)$ for Example 5.3.2.

We have shown above that the symmetric β-transformation gives a tiling if β is the golden mean (Example 5.3.4). Now we will see that this is not true if β is the Tribonacci number.

Example 5.4.1. Let β be the real solution of the equation $x^3 - x^2 - x - 1 = 0$. Let $I = A = \{-1, 0, 1\}$ and take $X_{\bar{1}} = \left[-\frac{1}{2}, -\frac{1}{2\beta}\right)$, $X_0 = \left[-\frac{1}{2\beta}, \frac{1}{2\beta}\right)$, $X_1 = \left[\frac{1}{2\beta}, \frac{1}{2}\right)$. See Figure 5.11 for the transformation and the natural extension domain $\widehat{X}$. The endpoints have the expansions

$$b(\ell_{\bar{1}}) = (\bar{1}001)^\omega,\ b(\ell_0) = 0(\bar{1}001)^\omega,\ b(\ell_1) = (1\bar{1}00)^\omega,$$

$$\tilde{b}(r_{\bar{1}}) = (\bar{1}100)^\omega,\ \tilde{b}(r_0) = 0(100\bar{1})^\omega,\ \tilde{b}(r_1) = (100\bar{1})^\omega.$$

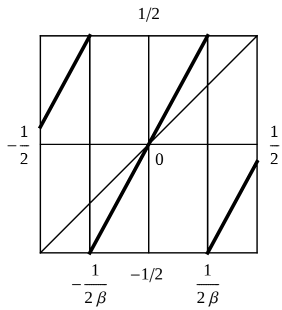
Therefore $\bar{\mathcal{S}}$ contains all sequences that do not contain any of the following forbidden finite sequences as a subsequence:

$$11, 101, 1000, 1001, \bar{1}\bar{1}, \bar{1}0\bar{1}, \bar{1}000, \bar{1}00\bar{1}.$$

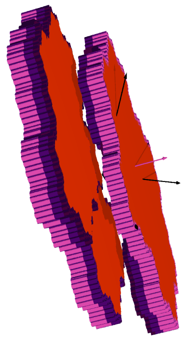
Then $\mathcal{S}$ is obtained from $\bar{\mathcal{S}}$ by excluding the suffix $(100\bar{1})^\omega$.

As for the golden mean, the support of the invariant measure is

$$\left[-\frac{1}{2}, \frac{\beta}{2} - 1\right) \cup \left[1 - \frac{\beta}{2}, \frac{1}{2}\right).$$



(a) This is T



(b) This is $\widehat{X}$

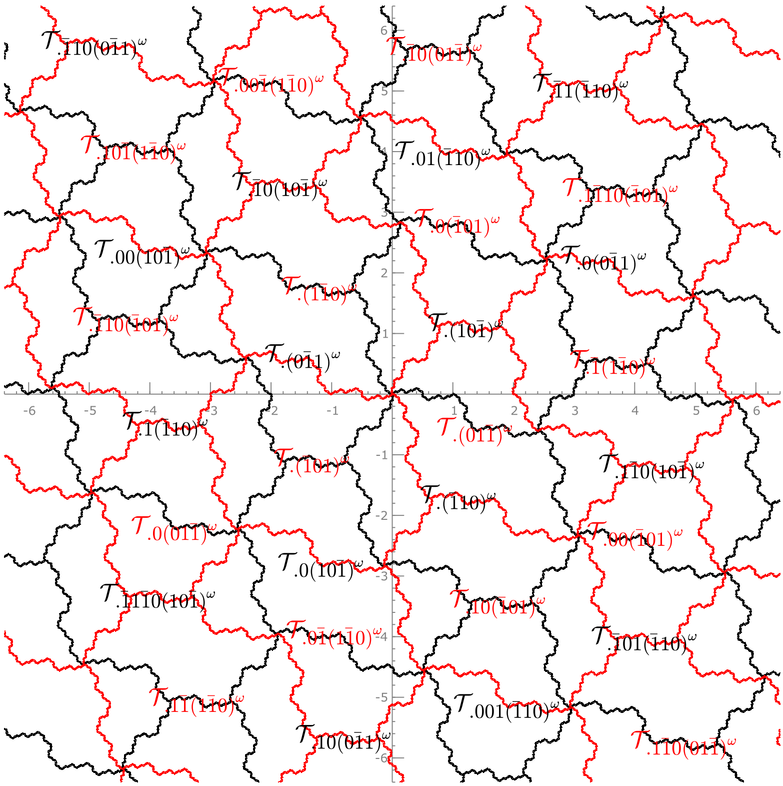


Figure 5.11 The double tiling for the symmetric β-transformation, with β the Tribonacci number.

In order to have a transformation where the support of its invariant measure is X, we therefore have to replace $\left[-\frac{1}{2}, \frac{1}{2}\right)$ by the two intervals $\left[-\frac{1}{2\beta}, \frac{\beta}{2}-1\right)$ and $\left[1-\frac{\beta}{2}, \frac{1}{2\beta}\right)$. For simplicity, we do not change the notation, and just exclude the sequence 000 in $\mathcal{S}$. So, we assume that any sequence in $\mathcal{S}$ can have at most two consecutive 0's in it.

Note that $\beta - 2 = -1/\beta^3$. It can be verified that $\mathcal{P}_1 = \big\{\pm 1/\beta^3, \pm 1/\beta^2, \pm(1 - 1/\beta)\big\}$. Let $z = \beta^2 - 1$, $\kappa = 7$. Then

$$\begin{aligned} T^\kappa(1/\beta^3 + z/\beta^\kappa) &= T^\kappa(1/\beta^2 + z/\beta^\kappa) = T^\kappa(1/\beta - 1 + z/\beta^\kappa) = 1/\beta - 1, \\ T^\kappa(-1/\beta^3 + z/\beta^\kappa) &= T^\kappa(-1/\beta^2 + z/\beta^\kappa) = T^\kappa(1 - 1/\beta + z/\beta^\kappa) = -1/\beta^3, \end{aligned}$$

thus $\Xi(\beta^2 - 1)$ lies only in $\mathcal{T}_{1/\beta-1}$ and $\mathcal{T}_{-1/\beta^3}$ by Corollary 5.4.3. Hence, the covering degree is at most 2. Showing that the covering degree is indeed 2 is more difficult than for Example 5.3.4 because the tiles have fractal boundary, see Figure 5.11.

We show with the help of the transducer in Figure 5.12 that every point in the subtile $M_\beta^5 \mathcal{T}_{\bullet\bar{1}0(10\bar{1})^\omega}$ of $\mathcal{T}_{\bullet(10\bar{1})^\omega}$ lies also in $\mathcal{T}_{\bullet(01\bar{1})^\omega}$. The nodes of the transducer have labels $(x; v)$, where $x = \bullet u$ stands for the real number $\bullet u0^\omega$ and v is given by the outputs of the incoming paths. For example, in $(.01; 001)$ we have $x = \frac{1}{\beta^2}$, and there exists an incoming path with edges $\xleftarrow{1|0}\xleftarrow{0|0}\xleftarrow{\bar{1}|1}$, which implies that the following output can only be $\bar{1}$. (The incoming path $\xleftarrow{1|0}\xleftarrow{0|1}$ imposes less restrictions on the following output.) For every transition $(x; v) \xrightarrow{i|i'} (x'; v')$, we have $x' = \frac{x + a_i - a'_i}{\beta}$. Consider a sequence of transitions starting at $(.01; 001)$,

$$(.01; 001) = (x_0; v_0) \xrightarrow{w_0|w'_0} (x_1; v_1) \xrightarrow{w_{-1}|w'_{-1}} (x_2; v_2) \xrightarrow{w_{-2}|w'_{-2}} \cdots$$

We have $.(10\bar{1})^\omega = 1 - 1/\beta$ and $.(01\bar{1})^\omega = 1/\beta^3$, thus $.(10\bar{1})^\omega - .(01\bar{1})^\omega = 1/\beta^2 = x_0$. By the construction of the transducer, we obtain $x_{k+1} = \bullet w_{-k} \cdots w_0 (10\bar{1})^\omega - \bullet w'_{-k} \cdots w'_0 (01\bar{1})^\omega$ for every $k \geq 0$. This means that

$$\varphi(w_{-k} \cdots w_0) + \Xi(.(10\bar{1})^\omega) = \varphi(w'_{-k} \cdots w'_0) + \Xi(.(01\bar{1})^\omega) + M_\beta^k \Xi(x_{k+1}).$$

Since x_{k+1} is bounded and M_β is contracting on H, we obtain therefore that $\varphi(\cdots w_{-1} w_0) + \Xi(.(10\bar{1})^\omega) \in \mathcal{T}_{\bullet(01\bar{1})^\omega}$ provided that $\cdots w'_{-1} w'_0 \cdot (01\bar{1})^\omega \in \mathcal{S}$. It can be easily verified that we always have $\cdots w'_{-1} w'_0 \cdot (01\bar{1})^\omega \in \mathcal{S}$ since the forbidden sequences given above are avoided.

For proving $M_\beta^5 \mathcal{T}_{\bullet\bar{1}0(10\bar{1})^\omega} \subseteq \mathcal{T}_{\bullet(01\bar{1})^\omega}$, it remains to show that every sequence $\bar{1}010\bar{1} w_{-5} w_{-6} \cdots$ satisfying

$$\cdots w_{-6} w_{-5} \bar{1}010\bar{1} \cdot (10\bar{1})^\omega \in \mathcal{S}$$

is the input of a path in the transducer. For a set of states Q, let $\bar{Q}$ denote the set of states that are obtained by replacing 1 by $\bar{1}$ and $\bar{1}$ by 1 in all the states from Q. The paths with input $\bar{1}010\bar{1}$ lead to the set of states

$$\bar{Q}_1 = \{(.\bar{1}0\bar{1}; 1), (.00\bar{1}; 0\bar{1}), (.0\bar{1}; 00\bar{1}), (.\bar{1}\bar{1}; 1)\}.$$

The paths from $\bar{Q}_1$ with input 1 lead to

$$Q_2 = \{(.101; \bar{1}), (.001; 01), (.00\bar{1}; 1), (.011; 00\bar{1})\},$$

the paths from $\bar{Q}_1$ with input 01 lead to

$$Q_1 = \{(.101; \bar{1}), (.001; 01), (.01; 001), (.11; \bar{1})\},$$

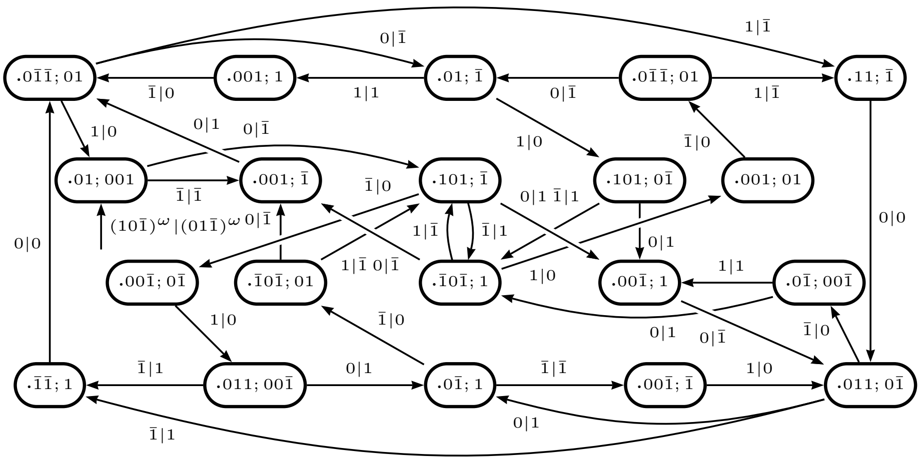


Figure 5.12 Double covering transducer for the symmetric β-transformation, $\beta^3 = \beta^2 + \beta + 1$.

and the paths from $\bar{Q}_1$ with input 001 lead to

$$Q_3 = \{(.01;001), (.11;\bar{1}), (.101;0\bar{1}), (.001;1)\}.$$

The paths from $\bar{Q}_2$ with input 1 lead to

$$Q_4 = \{(.101;\bar{1}), (.001;01), (.011;00\bar{1}), (.11;\bar{1})\},$$

the paths with input 01 lead to Q_3, as well as the paths with input 001. From $\bar{Q}_3$, 1 leads to

$$Q_5 = \{(.00\bar{1};1), (.101;\bar{1}), (.011;0\bar{1})\},$$

01 leads to Q_1, and 001 leads to Q_3. From $\bar{Q}_4$, 1 leads to Q_4, 01 leads to Q_3 and 001 leads to Q_3. From $\bar{Q}_5$, 1 leads to Q_1, 01 leads to Q_3 and 001 leads to Q_3. With the symmetry of the transducer and the description of the forbidden sequences in $\mathcal{S}$, this proves that every point in the subtile $M_\beta^5 \mathcal{T}_{\bullet\bar{1}0(10\bar{1})^\omega}$ of $\mathcal{T}_{\bullet(10\bar{1})^\omega}$ lies also in $\mathcal{T}_{\bullet(01\bar{1})^\omega}$. Since $\lambda^{d-1}(\mathcal{D}_{\bullet\bar{1}0(10\bar{1})^\omega}) > 0$, we have shown that the covering degree is 2.

Remember that it is conjectured that the greedy β-transformation $Tx = \beta x - \lfloor \beta x \rfloor$ on $X = [0,1)$ produces a tiling for every Pisot unit β. This is a version of the Pisot conjecture, see Conjecture 5.1.1. It is well known that it holds if β is the Tribonacci number, since $\mathcal{P}_1 = \{0\}$. It is therefore quite surprising that this conjecture does not hold when we move X from $[0,1)$ to $[-1/2, 1/2)$, and set $Tx = \beta x - \lfloor \beta x + 1/2 \rfloor$.

Everything from this chapter can also be found in [KS].

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SAMENVATTING

Getallen kunnen op veel verschillende manieren weergegeven worden. De meest gebruikelijke is de decimale ontwikkeling. Als we ergens 0,25 zien staan, dan interpreteren we dat als $\frac{1}{4}$. Oftewel, we lezen 0,25 = $\frac{2}{10} + \frac{5}{100}$. We kunnen ieder getal tussen 0 en 1 op een dergelijke manier schrijven, d.w.z. als een oneindige som van breuken met machten van 10 in de noemer en cijfers 0 tot en met 9 in de teller. Anders gezegd, voor iedere $x \in [0,1]$ is er een uitdrukking van de vorm $x = \sum_{k=1}^{\infty} \frac{b_k}{10^k}$, waarbij $b_k \in \{0,1,\dots,9\}$ voor alle $k \geq 1$. Deze uitdrukking heet de decimale ontwikkeling of decimale expansie van x.

De decimale expansie is een specifiek voorbeeld van een r-adische expansie. Met ieder geheel getal $r > 1$ is iedere $x \in [0,1]$ te schrijven als $x = \sum_{k=1}^{\infty} \frac{b_k}{r^k}$, waarbij $b_k \in \{0,1,\dots,r-1\}$ voor alle $k \geq 1$. De verzameling $\{0,\dots,r-1\}$ heet de cijferverzameling. Een manier om zo'n ontwikkeling te maken is m.b.v. een transformatie. Deze transformatie, T_r, is gedefinieerd van het interval $[0,1]$ naar zichzelf door $T_r 1 = 1$ en $T_r x = rx \pmod 1$ voor $0 \leq x < 1$. Anders gezegd, als $\frac{i}{r} \leq x < \frac{i+1}{r}$ voor een zekere i, dan is $T_r x = rx - i$. We krijgen de r-adische expansie van x door het interval $[0,1]$ in stukken te verdelen en de cijfers af te laten hangen van de positie van x en de punten $T_r^n x$ voor $n \geq 1$. Hier staat n voor de n-de iteratie, niet voor de n-de macht. Definieer $b_1(x) = i$ als $x \in \left[\frac{i}{r}, \frac{i+1}{r}\right)$, $0 \leq i \leq r-1$, en $b_1(1) = r-1$. Voor $n \geq 1$, laat $b_n(x) = b_1(T_r^{n-1}x)$. Dan geldt $T_r x = rx - b_1(x)$ en dus kunnen we voor iedere $n \geq 1$ schrijven dat

$$x = \frac{b_1(x)}{r} + \frac{T_r x}{r} = \frac{b_1(x)}{r} + \frac{b_2(x)}{r^2} + \frac{T_r^2 x}{r^2} = \cdots = \sum_{k=1}^{n} \frac{b_k(x)}{r^k} + \frac{T_r^n x}{r^n}.$$

Aangezien $T_r^n x \in [0,1]$ voor iedere $n \geq 0$, krijgen we dat $x = \sum_{k=1}^{\infty} \frac{b_k(x)}{r^k}$. In het volgende plaatje is te zien hoe de eerste vijf cijfers van de ternaire ontwikkeling van $\frac{1}{2}\sqrt{3}$ zo verkregen worden. We zien dat

$$\frac{1}{2}\sqrt{3} = \frac{2}{3} + \frac{1}{9} + \frac{2}{27} + \frac{1}{81} + \frac{0}{243} + \cdots .$$

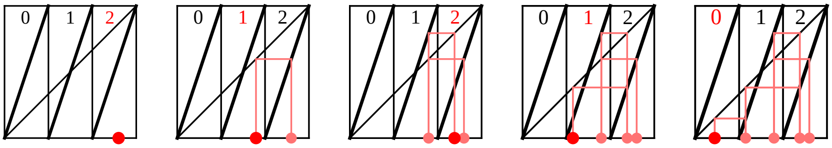


Figure 13 De eerste vijf cijfers van de ternaire ontwikkeling van $\frac{1}{2}\sqrt{3}$ zijn 21210.

De iteraties van de transformatie T_r geven dus cijfers van r-adische ontwikkelingen. Het voordeel van het hebben van een dergelijke transformatie is dat we de ontwikkelingen kunnen bestuderen door naar de transformatie te kijken. We kunnen ergodentheorie gebruiken om eigenschappen van de ontwikkelingen te geven. In ergodentheorie wordt het lange termijn gedrag van dynamische systemen bestudeerd. Onder een dynamisch systeem wordt hier een viertal $(X, \mathcal{F}, \mu, T)$ verstaan, waarin $(X, \mathcal{F}, \mu)$ een kansruimte is en $T : X \to X$ een meetbare transformatie, zodat $\mu(T^{-1}E) = \mu(E)$ voor alle verzamelingen $E \in \mathcal{F}$. Een maat μ met deze laatste eigenschap heet invariant m.b.t. de transformatie T. Een transformatie heet ergodisch als het dynamische systeem niet op te splitsen is in kleinere stukken, die onder de transformatie niet met elkaar in aanraking komen. De transformatie T_r vormt samen met de kansruimte $([0,1], \mathcal{B}, \lambda)$ een ergodisch dynamisch systeem, waarbij $\mathcal{B}$ de Borel σ-algebra is op $[0,1]$ en λ de Lebesgue maat op $([0,1], \mathcal{B})$.

De dynamica van de r-adische transformatie is vrij eenvoudig. Een belangrijke eigenschap van r-adische expansies is dat bijna alle getallen een unieke r-adische ontwikkeling hebben en dat de getallen die geen unieke ontwikkeling hebben, precies twee ontwikkelingen hebben. Bedenk bijvoorbeeld dat

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \cdots \quad \text{en} \quad \frac{1}{4} = \frac{2}{10} + \frac{4}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots .$$

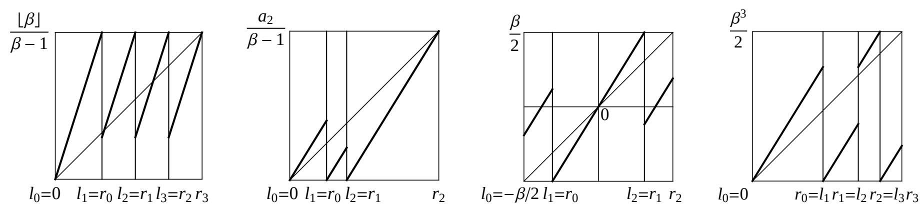
Deze eigenschap verandert drastisch als we getallen gaan ontwikkelen met niet-gehele basis. Stel we nemen een niet-geheel getal $\beta > 1$ als basis en een cijferverzameling $A = \{a_0, \cdots, a_m\}$, die bestaat uit eindig veel willekeurige reële getallen. Als β en A aan de volgende eigenschappen voldoen,

- $a_0 < a_1 < \cdots < a_m$,
- $\max\limits_{0 \le i \le m-1} (a_{i+1} - a_i) \le \dfrac{a_m - a_0}{\beta - 1}$,

dan kan iedere $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ geschreven worden als $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$, met $b_k \in A$ voor alle $k \geq 1$. Zo'n uitdrukking heet een β-expansie van x met cijfers in A of een β-expansie van x met willekeurige cijfers. Een cijferverzameling met deze twee eigenschappen heet toelaatbaar voor β. Het aantal cijfers in een toelaatbare cijferverzameling is minimaal gelijk aan het kleinste gehele getal groter of gelijk aan β. In het algemeen geldt dat met een toelaatbare A bijna alle getallen oneindig veel verschillende β-expansies met cijfers in A hebben.

Daarom is er ook niet een enkele transformatie die ze genereert, maar zijn er veel verschillende. De twee belangrijkste transformaties zijn de gulzige en de luie transformatie. De gulzige transformatie produceert β-expansies die op iedere plek het grootst mogelijke cijfer uit A hebben en de luie transformatie doet precies het tegenovergestelde. Die zet op iedere plek het kleinst mogelijke cijfer uit A. Deze twee transformaties zijn isomorf, de ene transformatie kan op een makkelijke manier uit de andere verkregen worden. De gulzige transformatie heeft een unieke invariante maat die absoluut continu is m.b.t. de Lebesguemaat. De transformatie is ergodisch m.b.t. deze invariante maat en de verzameling waarop de dichtheid van deze maat positief is, is een interval. Door het isomorfisme geldt hetzelfde voor de luie transformatie.

In bepaalde situaties is er een expliciete uitdrukking voor de dichtheid van de invariante maat, bijvoorbeeld als het aantal cijfers in de cijferverzameling minimaal is of als het dynamisch systeem gegeven kan worden door een Markovketen. Als de cijferverzameling het minimale aantal cijfers bevat is de gulzige transformatie zwak Bernoulli. Voor alle toelaatbare cijferverzamelingen met drie cijfers kan een expliciete uitdrukking voor de dichtheid verkregen worden door de natuurlijke uitbreiding. Een natuurlijke uitbreiding van een niet-inverteerbaar systeem is een inverteerbaar dynamisch systeem, waarin de dynamische eigenschappen van het originele systeem terug te vinden zijn en het is het kleinste systeem, in maattheoretische zin, met deze eigenschappen. In het tweede plaatje van de afbeelding hieronder is een gulzige transformatie te zien. Uit het plaatje blijkt al dat een dergelijke transformatie in het algemeen niet inverteerbaar is. De natuurlijke uitbreiding van de gulzige transformatie bestaat uit een aantal rechthoeken, waartussen een transformatie is gedefinieerd. De Lebesguemaat is de invariante maat van de natuurlijke uitbreiding en de invariante maat voor de bijbehorende gulzige transformatie kan gevonden worden door projectie op de eerste coördinaat. Met deze invariante maat kan aangetoond worden dat de gulzige transformatie exact is.



(a) $\beta = \pi$ en $A = \{0, 1, 2, 3\}$

(b) $\beta = \frac{1+\sqrt{5}}{2}$ en $A = \{0, 2\beta, 5\}$

(c) $\beta = \frac{1+\sqrt{5}}{2}$ en $A = \{-1, 0, 1\}$

(d) $\beta = \frac{1+\sqrt{5}}{2}$ en $A = \{0, \beta, \frac{\beta}{2}, \frac{3\beta+1}{2}\}$

Figure 14 Voorbeelden van transformaties die β-expansies met cijfers in A genereren.

M.b.v. de gulzige en luie transformatie kunnen we een hele familie transformaties maken, die allemaal β-expansies met willekeurige cijfers genereren. Het plaatje hierboven geeft wat voorbeelden van dergelijke transformaties.

Ook is er een transformatie die expansies genereert d.m.v. een stochastisch proces. Deze transformatie bepaalt het cijfer op ieder tijdstip met het gooien van een dobbelsteen en is isomorf met een Bernoulli shift. Als $x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}$ met b_k in een toelaatbare cijferverzameling A voor alle $k \geq 1$, dan kan deze expansie geïdentificeerd worden met de rij getallen $b_1 b_2 \cdots$, de cijferrij. Er is een manier om voor een een transformatie te bepalen welke cijferrijen erdoor voortgebracht worden.

Voor β's met bijzondere eigenschappen kunnen we meer zeggen. Stel dat β een Pisot eenheid is, d.w.z. dat $\beta > 1$ een reële oplossing is van een vergelijking van de vorm $x^d - c_1 x^{d-1} - \cdots - c_{d-1}x - c_d = 0$ met $|c_d| = 1$, $c_k \in \mathbb{Z}$ voor alle $1 \leq k \leq d-1$ en zo dat alle andere oplossingen van deze vergelijking in modulus kleiner zijn dan 1. Als de cijferverzameling A alleen elementen uit $\mathbb{Q}(\beta)$ bevat, dan is er voor een grote klasse transformaties die β-expansies met cijfers in A voortbrengen, een d-dimensionale natuurlijke uitbreiding, die de Lebesguemaat als invariant maat heeft. M.b.v. deze natuurlijke uitbreiding kunnen we een meervoudige betegeling geven van een ruimte H die geïdentificeerd kan worden met $\mathbb{R}^{d-1}$. Meervoudige betegeling wil zeggen dat er een constante $\textsc{m} \geq 1$ is zodat bijna alle punten van H in precies $\textsc{m}$ tegels liggen. Deze constante heet de overdekkingsgraad. Een meervoudige betegeling met overdekkingsgraad 1 is een betegeling. De tegels van deze meervoudige betegeling worden gegeven door alle mogelijke verledens van de punten uit $\mathbb{Z}[\beta]$ die in het domein van de transformatie liggen. De meervoudige betegeling geeft eigenschappen van de expansies. Zo ligt de oorsprong van H precies in die tegels die horen bij punten waarvan de cijferrij puur periodiek is. Er zijn manieren om na te gaan wat de overdekkingsgraad is van een meervoudige betegeling. Deze overdekkingsgraad is bijvoorbeeld 1 dan en slechts dan als de natuurlijke uitbreiding een betegeling geeft van de torus. Voor de gulzige transformatie met cijfers in $\{0, 1, \ldots, \lfloor \beta \rfloor\}$, waarbij $\lfloor \beta \rfloor$ het grootste gehele getal niet groter dan β is, bestaat het vermoeden dat deze meervoudige betegeling altijd overdekkingsgraad 1 heeft, als β een Pisot getal is. Dit vermoeden heet het Pisotvermoeden. In de grotere klasse van β-transformaties met willekeurige cijfers zijn voorbeelden te vinden waarvoor de meervoudige betegeling ook echt meervoudig is, dus overdekkingsgraad groter dan 1 heeft.

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CURRICULUM VITAE

Charlene Kalle was born on December 11, 1979 in Bergen op Zoom. From 1992 to 1998 she attended the Rooms Katholiek Gymnasium Juvenaat Heilig Hart in Bergen op Zoom.

In 1999 she started studying mathematics at Utrecht University and in 2000 she started studying Italian language and culture at Utrecht University as well. In 2002 she co-authored a book on mathematics and the Islam for high school students. In 2003 she took part in the Erasmus exchange program and spent six months at Università La Sapienza in Rome. In 2005 she obtained a Master's degree in mathematics with a minor in philosophy cum laude. The results of the Master thesis, entitled *Convergence of logarithmic return times*, were later published in an article. In 2005 she also obtained a Bachelor's degree in Italian language and culture with a minor in Spanish language and culture cum laude.

From 2005 to 2009 she did her Ph.D. research under the supervision of Dr. Karma Dajani at Utrecht University, which resulted in a number of articles. These results can all be found in this thesis. During this period she visited several conferences, workshops and summer schools and reported on her research on various occasions. She was a regional coordinator for the European Women in Mathematics and co-organized an EWM-day in Amsterdam in 2008. In the last 18 months she made several visits to Paris to work with Dr. Wolfgang Steiner from Université Paris Diderot - Paris 7, which led to the results from Chapter 5. During the last year she was one of the editors of the Dutch mathematics magazine Nieuw Archief voor Wiskunde.

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