A NATURAL EXTENSION FOR THE GREEDY β-TRANSFORMATION WITH THREE ARBITRARY DIGITS

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Abstract. We construct a planar version of the natural extension of the piecewise linear transformation T generating greedy β-expansions with digits in an arbitrary set of real numbers $A = \{a_0, a_1, a_2\}$. As a result, we derive in an easy way a closed formula for the density of the unique T-invariant measure μ absolutely continuous with respect to Lebesgue measure. Furthermore, we show that T is exact and weak Bernoulli with respect to μ.

1. Introduction

Let $\beta > 1$ be a real number, and $A = \{a_0, a_1, \ldots, a_m\}$ a set of real numbers. An expression of the form

$$(1) \qquad x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n},$$

where $b_n \in A$ is called a β-expansion of x with digits in the set A. In [12], Pedicini showed that if $\beta > 1$, and if the set of real numbers $A = \{a_0, a_1, \ldots, a_m\}$ satisfies the following two conditions:

(i) $a_0 < a_1 < \cdots < a_m$,

(ii) $\max_{1 \leq j \leq m}(a_j - a_{j-1}) \leqq \frac{a_m - a_0}{\beta - 1}$,

then every $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1} \right]$ has an expansion of the form (1), with $b_n \in A$ for all $n \geqq 1$. He also gave a recursive algorithm that generates *greedy* expansions of the form (1), in the sense that at each step of the algorithm, b_n is chosen to be the largest element of A such that

$$(2) \qquad \sum_{i=1}^{n} \frac{b_i}{\beta^i} + \sum_{i=n+1}^{\infty} \frac{a_1}{\beta^i} \leqq x.$$

We call a set A, satisfying (i) and (ii) an *allowable digit set*. In [3] it was shown that if A is allowable, then one can construct a dynamical system whose iterates generate, for all points $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1} \right]$, all possible expansions of the form (1). If A is not allowable, then not every point has an expansion. The size of the set of real numbers that can be represented by the expression (1), for different choices of β and for $A = \{0, 1, 3\}$, was studied by Keane, Smorodinsky and Solomyak in [9]. Later, Pollicott and Simon [13] generalized their results to the case A is a set consisting of non-negative integers. They gave the Hausdorff dimension of the set of points that possess an expansion of the form (1).

Throughout this paper, we will assume that A is an allowable set. In case $A = \left\{ 0, 1, \ldots, \lfloor \beta \rfloor \right\}$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, then one is led to the classical case. The interest in such expansions was initiated by Rényi [14], in his introduction of β-expansions generated by iterating the map

$$T_\beta : \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1} \right] \to \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1} \right] : x \to \begin{cases} \beta x \ (\text{mod } 1), & \text{if } \ 0 \leqq x < \dfrac{\lfloor \beta \rfloor}{\beta}, \\[2mm] \beta x - \lfloor \beta \rfloor, & \text{if } \ \dfrac{\lfloor \beta \rfloor}{\beta} \leqq x \leqq \dfrac{\lfloor \beta \rfloor}{\beta - 1}. \end{cases}$$

If we set

$$(3) \qquad b_1^c = b_1^c(x) = \begin{cases} i, & \text{if } \ x \in \left[\dfrac{i-1}{\beta}, \dfrac{i}{\beta} \right), \ \text{for } i = 1, \ldots, \lfloor \beta \rfloor, \\[3mm] \lfloor \beta \rfloor, & \text{if } \ x \in \left[\dfrac{\lfloor \beta \rfloor}{\beta}, \dfrac{\lfloor \beta \rfloor}{\beta - 1} \right], \end{cases}$$

and for $n \geqq 1$, $b_n^c = b_n^c(x) = b_1^c(T_\beta^n x)$, then $T_\beta x = \beta x - b_1^c$, and for any $n \geqq 1$,

$$x = \sum_{i=1}^{n} \frac{b_i^c}{\beta^i} + \frac{T_\beta^n x}{\beta^n}.$$

Letting $n \to \infty$, it is easily seen that $x = \sum_{n=1}^{\infty} \frac{b_n^c}{\beta^n}$. We call the β-expansion generated by the transformation T_β the *classical greedy* β-expansion of x. More precisely, for each $n \geq 1$, if $b_1^c, \ldots, b_{n-1}^c$ are already known, then b_n^c is the largest element of the complete digit set $A = \{0, 1, \ldots \lfloor \beta \rfloor\}$, such that

$$\sum_{i=1}^{n} \frac{b_i^c}{\beta^i} \leq x.$$

The map T_β gives one algorithm that generates (classical) β-expansions of the form (1) with digits in $A = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. In fact, a Lebesgue a.e. point in $\left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$ has uncountably many possible expansions of the form (1) (see [17], [5], and [6]).

Let λ denote the 1-dimensional Lebesgue measure. The transformation T_β has a unique invariant measure, absolutely continuous with respect to λ. Rényi proved the existence of this measure in [14], and Gel'fond and Parry independently, gave an explicit formula for the density function of this measure in [7] and [11] respectively. The invariant measure has the unit interval $[0, 1)$ as its support and the density function h_c is given as

$$(4) \qquad h_c : [0, 1) \to [0, 1) : x \mapsto \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n 1)}(x),$$

where $F(\beta) = \int_0^1 \sum_{x < T_\beta^n 1} \frac{1}{\beta^n} d\lambda$ is a normalizing constant. From now on we will refer to an invariant measure, absolutely continuous with respect to λ as an *ACIM*.

Let $\beta > 1$, and $A = \{a_0, \ldots, a_m\}$ any allowable digit set. In [3], a transformation $T_{\beta,A}$ is given whose iterates generate greedy β-expansions of the form (1), and satisfying (2). It is also shown that $T_{\beta,A}$ is conjugate to $T_{\beta,\tilde{A}}$, where $\tilde{A} = \{0, a_1 - a_0, \ldots, a_m - a_0\}$, i.e. a digit set of which the first digit equals zero. Therefore, without loss of generality, and for ease of notation, we shall assume that $a_0 = 0$, i.e. $A = \{0, a_1, \ldots, a_m\}$. In this case, the underlying transformation $T_{\beta,A} = T : \left[0, \frac{a_m}{\beta - 1}\right] \to \left[0, \frac{a_m}{\beta - 1}\right]$ is given by

$$Tx = \begin{cases} \beta x - a_j, & \text{if } x \in \left[\dfrac{a_j}{\beta}, \dfrac{a_{j+1}}{\beta}\right), \text{ for } j = 0, \ldots, m-1, \\[3mm] \beta x - a_m, & \text{if } x \in \left[\dfrac{a_m}{\beta}, \dfrac{a_m}{\beta - 1}\right]. \end{cases}$$

The sequence of digits $\{b_n\}_{n \geq 1}$ can be defined in a way similar to (3) as follows. Set $b_1(x) = b_1 = a_j$ if $x \in \left[\frac{a_j}{\beta}, \frac{a_{j+1}}{\beta}\right)$, and $b_1(x) = b_1 = a_m$ if

$x \in \left[\frac{a_m}{\beta}, \frac{a_m}{\beta - 1} \right]$. Let $b_n = b_n(x) = b_1(T^{n-1} x)$ for $n \geqq 1$. One easily sees that $x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$ with b_n satisfying (2), i.e. T generates greedy β-expansions with digits belonging to the set A. We refer to $T = T_{\beta, A}$ as the greedy β-transformation with digit set A (see [3] for more information). If $x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$ is the greedy β-expansion of x, we also write

$$x =_\beta b_1 b_2 b_3 \dots,$$

which is understood to mean the same as (1).

From [2] we know that the transformation T has an ACIM that is unique and ergodic. The support of this measure is given by the interval $[0, a_{i_0} - a_{i_0 - 1})$, where

$$(5) \quad i_0 = \min \left\{ i \in \{1, \dots, m\} : T[0, a_i - a_{i-1}) \subseteqq [0, a_i - a_{i-1}) \ \lambda \text{ a.e. } \right\}.$$

There are some results on formulas for densities of general piecewise linear maps. In particular, Kopf [10] considered a class of piecewise linear, expanding maps from the interval $[0, 1]$ to itself, that leave the points 0 and 1 fixed. He constructed a matrix M, the entries of which consist of infinite sums of indicator functions, and he used a vector from the nullspace of M to obtain the density function. A more recent result can be found in [8] from Góra. He considered an even more general class of piecewise linear maps. In his setting, the maps only have to be eventually expanding, which means that for each slope β_i there must exist an $n \geqq 1$ such that $|\beta_i|^n > 1$. The slopes can also be negative, under the same condition. For this class of transformations, Góra constructed a matrix S and used the solutions of a certain linear system involving S to obtain the density function. Two main differences between their two methods are the following. First of all, Kopf makes the extra assumption that the points 0 and 1 are fixed. More importantly, Kopf obtains all invariant densities, while Góra gives only one version of the density for each ergodic component. We seek a form similar to that given by Wilkinson [18], see (10) ahead. In [2], it is shown that if

$$(6) \qquad\qquad\qquad m < \beta \leqq m + 1,$$

then indeed the density of the ACIM is given by (10) (notice that $m + 1$ is equal to the number of digits).

In [3] it is proven that the minimal amount of digits in an allowable digit set is $\lceil \beta \rceil$. In other words, the amount of digits in A is at least equal to the smallest integer larger than or equal to β.

Let N be the largest element of the set $\{1, \dots, m\}$ such that $\frac{a_N}{\beta} < a_{i_0} - a_{i_0 - 1}$. Define a partition $\Delta = \left\{ \Delta(a_i) : 0 \leqq i \leqq N \right\}$ of the support of the ACIM of T, where for $i = 0, \dots, N - 1$, we have

$$\Delta(a_i) = \left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta} \right), \quad \text{and} \quad \Delta(a_N) = \left[\frac{a_N}{\beta}, a_{i_0} - a_{i_0 - 1} \right).$$

Note that $T\Delta(a_N) = \big[0, T(a_{i_0} - a_{i_0-1})\big)$ and for $i \in \{0, \ldots, N-1\}$, $T\Delta(a_i) = [0, a_{i+i} - a_i)$. Using Δ and T, we can make a sequence of partitions $\big\{\Delta^{(n)}\big\}$: for $n \geqq 0$,

$$(7) \qquad \Delta^{(n)} = \bigvee_{k=0}^{n-1} T^{-k}\Delta.$$

The elements of $\Delta^{(n)}$ are intervals and are called the *fundamental intervals of rank n*. If

$$\Delta(b_0) \cap T^{-1}\Delta(b_1) \cap \ldots \cap T^{-(n-1)}\Delta(b_{n-1})$$

is an element of $\Delta^{(n)}$, denote it by $\Delta(b_0 b_1 \ldots b_{n-1})$. We call a fundamental interval $\Delta(b_0 b_1 \ldots b_{n-1})$ *full* of rank n if

$$\lambda\big(T^n\Delta(b_0 b_1 \ldots b_n)\big) = a_{i_0} - a_{i_0-1}$$

and *non-full* otherwise. This means that for a full fundamental interval, $\Delta(b_0 \ldots b_{n-1})$ we have

$$(8) \qquad \lambda\big(\Delta(b_0 b_1 \ldots b_{n-1})\big) = \frac{a_{i_0} - a_{i_0-1}}{\beta^n}$$

and if $\Delta(b_0 \ldots b_{n-1})$ is non-full, then

$$(9) \qquad \lambda\big(\Delta(b_0 b_1 \ldots b_{n-1})\big) < \frac{a_{i_0} - a_{i_0-1}}{\beta^n}.$$

Let B_n be the collection of all non-full fundamental intervals of rank n, that are not subsets of any full fundamental interval of lower rank. For $x \in [0, a_{i_0} - a_{i_0-1})$, define $\phi_0(x) = 1$ and for $n \geqq 1$, let

$$\phi_n(x) = \sum_{\Delta(b_0 b_1 \ldots b_{n-1}) \in B_n} \frac{1}{\beta^n} 1_{T^n\Delta(b_0 b_1 \ldots b_{n-1})}(x).$$

Put $\phi = \sum_{n=0}^{\infty} \phi_n$. In [3], it is shown that for $m < \beta \leqq m+1$, ϕ is integrable and the function

$$(10) \qquad h : [0, a_{i_0} - a_{i_0-1}) \to [0, a_{i_0} - a_{i_0-1}) : x \mapsto \frac{\phi(x)}{\int \phi(x) d\lambda(x)}$$

is the density function of the ACIM of T. This density is in fact a special case of the one found by Wilkinson in [18] for a special class of piecewise linear

and expanding maps. Notice that for the classical greedy β-transformation, B_n contains at most one element $\Delta(b_0 \ldots b_{n-1})$ and for this element we have

$$T_\beta^n \Delta(b_0 \ldots b_{n-1}) = [0, T_\beta^n 1).$$

So the density function from (4) is a special case of the density function from (10).

In this article we give a natural extension of the greedy β-transformation with three digits, i.e. with an allowable digit set of the form $\{0, a_1, a_2\}$. In general, a natural extension is the smallest invertible dynamical system (in the measure theoretic sense), that contains the dynamics of the original transformation as a subsystem. The concept of a natural extension of a non-invertible system was introduced by Rohlin [15], where he gave a canonical way of constructing a natural extension, showed that it is unique up to isomorphism, and proved that it possesses similar dynamical properties as the original system. This is an interesting object, since it often demonstrates a dynamics that is easier to understand than the original non-invertible system and it can also be used to derive properties of the original system. There are many ways to construct the natural extension, each having its own merit. As a result of our construction for the greedy β-transformation, we show that the density given in (10) holds for any allowable digit set $A = \{0, a_1, a_2\}$. Since the minimal amount of digits in an allowable digit set is $\lceil \beta \rceil$, one sees that in this case $1 < \beta < 3$. Every β-transformation with digit set $A = \{0, a_1, a_2\}$ is isomorphic to a greedy β-transformation with digit set $\{0, 1, u\}$, where $u = \frac{a_2}{a_1}$. Throughout the rest of the paper, we will assume that $1 < \beta < 3$, and $A = \{0, 1, u\}$ is an allowable set. Our construction resembles the version built in [4] for the classical greedy β-transformation (see also [1]). The domain of the natural extension is roughly a (union) of rectangular regions in $\mathbb{R}^2$, with invariant measure the restriction of the 2-dimensional Lebesgue measure to our domain. The projection of the invariant measure in the first coordinate gives the desired invariant density of T of the form given in (10). We also show that under the invariant measure obtained, T is exact and weakly Bernoulli. To illustrate the construction of our version of the natural extension, an example of a specific greedy β-transformation with three digits can be found in the last section. Here β is the golden mean and $A = \left\{ 0, 1, \frac{4}{3} \right\}$.

2. A closer look at the greedy β-transformation with three digits

Let $1 < \beta < 3$, and $A = \{0, 1, u\}$ an allowable digit set. The corresponding greedy β-transformation $T = T_{\beta,A}$ has the form

$$Tx = \begin{cases} \beta x, & \text{if } x \in \left[0, \dfrac{1}{\beta}\right), \\[2ex] \beta x - 1, & \text{if } x \in \left[\dfrac{1}{\beta}, \dfrac{u}{\beta}\right), \\[2ex] \beta x - u, & \text{if } x \in \left[\dfrac{u}{\beta}, \dfrac{u}{\beta - 1}\right). \end{cases}$$

The sequence of digits $\{b_n\}_{n \geq 1}$ is defined for any $x \in \left[0, \frac{u}{\beta-1}\right]$ as in the introduction. From (5), we have that the support of the ACIM is either the interval $[0, 1)$, or the interval $[0, u - 1)$. The following situations can occur.

Suppose first that $\beta < u$, then either $T1 \leqq 1$ or $T1 > 1$.

If $T1 \leqq 1$, then the support is $[0, 1)$ and since $T1 = \beta - 1$, we get that $\beta \leqq 2$. The transformation T on the interval $[0, 1)$ is then isomorphic to the classical greedy β-transformation, T_β. Figure 1(a) is an example of this.

(a) $\beta = \sqrt{3}$ and $A = \{0, 1, 3\}$

(b) $\beta = 1 + \sqrt{2}$ and $A = \{0, 1, 3\}$

(c) $\beta = \sqrt{7}$ and $A = \{0, 1, \frac{7}{3}\}$

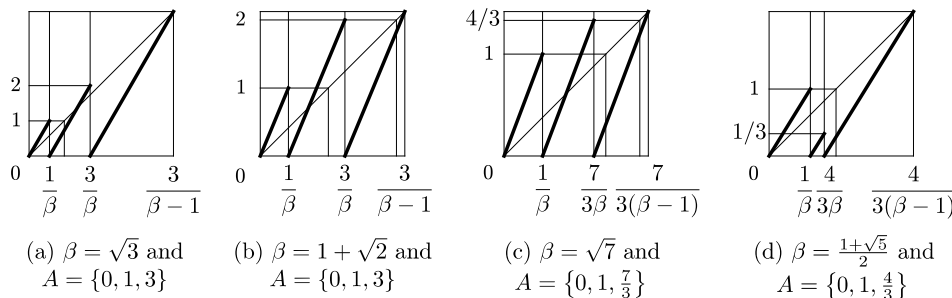(d) $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{0, 1, \frac{4}{3}\}$

Fig. 1: *Examples of the four possibilities for the support of the* ACIM *of the greedy β-transformation with three digits*

If $T1 > 1$, then the support of the invariant measure is the other interval, $[0, u - 1)$, and we can deduce that $\beta > 2$. So, criterion (6) applies, and the density for the invariant measure is given by (10). In Figure 1(b) we see an example.

Suppose that $u \leqq \beta$. Either $u > 2$ or $u \leqq 2$.

If $u > 2$, then the support of the ACIM is $[0, u - 1)$ and we have that $2 < u \leqq \beta$. So again $\beta > 2$ and the density from equation (10) is the density of the ACIM. See Figure 1(c) for an example.

If $u = 2$, then the support is $[0, 1) = [0, u - 1)$ and $\beta \geqq 2$. The transformation in this case is isomorphic to the classical greedy β-transformation T_β. Similarly, if $u = \beta < 2$, then the support of the ACIM is $[0, 1)$. The transformation T is again isomorphic to T_β.

Lastly, suppose that $u = \beta - 1$. Then it holds that $\beta > 2$, since $u > 1$ and thus the density from (10) is the density for the ACIM.

The only situation we did not yet consider, is when $u < \min \{\beta, 2\}$ and $u \neq \beta - 1$. To this, the rest of this article is dedicated. Figure 1(d) gives an example of a transformation that satisfies these conditions.

From now on, suppose that T is a greedy β-transformation with an allowable digit set $A = \{0, 1, u\}$, that satisfies the following conditions: $u < \min \{\beta, 2\}$ and $u \neq \beta - 1$. Of course, since A is allowable and $u \neq \beta - 1$, we also have $u > \beta - 1$. This is all captured in the following condition:

(11) $$\max \{\beta - 1, 1\} < u < \min \{2, \beta\}.$$

Notice that we do not assume that $\beta \leqq 2$, although we already know that the density from (10) is the density of the ACIM in this case. The reason is that the construction of the natural extension that will be given in what follows, is also valid for $2 < \beta < 3$.

REMARK 2.1. Observe that, if $1 < \beta \leqq 2$, then $u < \beta$ implies condition (11). So, if $1 < \beta \leqq 2$ and a digit set $A = \{0, 1, u\}$ satisfies $u < \beta$, then A is an allowable digit set.

The support of the ACIM of the transformation T is the interval $[0, 1)$. The partition $\Delta = \big\{ \Delta(0), \Delta(1), \Delta(u) \big\}$ of this interval is given in the following way:

$$\Delta(0) = \left[0, \frac{1}{\beta}\right), \quad \Delta(1) = \left[\frac{1}{\beta}, \frac{u}{\beta}\right), \quad \Delta(u) = \left[\frac{u}{\beta}, 1\right).$$

As explained in the introduction, we can construct the sequence of partitions $\big\{\Delta^{(n)}\big\}_{n \geqq 1}$, with $\Delta^{(n)}$ as defined in (7). The elements of $\Delta^{(n)}$ are denoted by $\Delta(b_0 \ldots b_{n-1})$ and are either full or non-full fundamental intervals of rank n. We have the following obvious lemma.

LEMMA 2.1. *Let $\Delta(d_0 \ldots d_{p-1})$ and $\Delta(e_0 \ldots e_{q-1})$ be two full fundamental intervals of rank p and q respectively. Then $\Delta(d_0 \ldots d_{p-1} e_0 \ldots e_{q-1})$ is a full fundamental interval of rank $p + q$.*

Recall that for $n \geqq 1$, B_n is the collection of all non-full fundamental intervals of rank n that are not contained in any full fundamental interval of lower rank. Let $\kappa(n)$ be the number of elements in B_n. So $\kappa(1) = 2$, since this is the number of non-full fundamental intervals of rank 1 and for all $n \geqq 1$, $\kappa(n) \leqq 2^n$. The version of the natural extension that we will define in the next section, uses all the elements of B_n for all $n \geqq 1$. To make sure that the total measure of the underlying space of this version is finite, we need upper bounds for the values of $\kappa(n)$. To obtain these, we will first describe the structure of the elements of B_n.

Notice that by (9) we have that for all elements $\Delta(b_0 \ldots b_{n-1}) \in B_n$,

(12) $$\lambda\big(\Delta(b_0 \ldots b_{n-1})\big) < \frac{1}{\beta^n}.$$

For an $x \in [0, 1)$, the set $\{T^n x : n \geqq 0\}$ is called the *orbit* of x under the transformation T. Let

$$u - 1 = \sum_{n=1}^{\infty} \frac{d_n^{(1)}}{\beta^n} =_\beta d_1^{(1)} d_2^{(1)} d_3^{(1)} \ldots, \quad \beta - u = \sum_{n=1}^{\infty} \frac{d_n^{(2)}}{\beta^{n+1}} =_\beta d_1^{(2)} d_2^{(2)} d_3^{(2)} \ldots,$$

be the greedy β-expansions with digit set A of the points $u - 1$ and $\beta - u$, i.e. the expansions generated by iterations of T. The number $\beta - u$ would be the image of 1 under T if T were not restricted to the interval $[0, 1)$. The values of the numbers $\kappa(n)$ depend on the orbits of the points $\beta - u$ and $u - 1$ under T and whether or not $T^i(u - 1)$ and $T^i(\beta - u)$ are elements of $\Delta(u)$ for $0 \leqq i < n$. To see this, notice that for any $\Delta(b_0 \ldots b_{n-1}) \in B_n$ one has $b_0 \in \{1, u\}$ and the set $T^n \Delta(b_0 \ldots b_{n-1})$ has the form

$$\big[0, T^i(u - 1) \big) \ \text{or} \ \big[0, T^i(\beta - u) \big)$$

for some $0 \leqq i < n$. Suppose $T^n \Delta(b_0 \ldots b_{n-1}) = \big[0, T^i(u - 1) \big)$.

If $\lambda \big(\big[0, T^i(u - 1) \big) \cap \Delta(u) \big) = 0$, then $\Delta(b_0 \ldots b_{n-1})$ contains exactly one element of B_{n+1}, namely $\Delta(b_0 \ldots b_{n-1} 0)$ in case $\lambda \big(\big[0, T^i(u - 1) \big) \cap \Delta(1) \big) = 0$ or $\Delta(b_0 \ldots b_{n-1} 1)$ in case $\lambda \big(\big[0, T^i(u - 1) \big) \cap \Delta(1) \big) > 0$. Furthermore, in the first case,

$$T^{n+1} \Delta(b_0 \ldots b_{n-1} 0) = \big[0, T^{i+1}(u - 1) \big)$$

and also in the second case,

$$T^{n+1} \Delta(b_0 \ldots b_{n-1} 1) = \big[0, T^{i+1}(u - 1) \big).$$

On the other hand, if $\lambda \big(\big[0, T^i(u - 1) \big) \cap \Delta(u) \big) > 0$, then $\Delta(b_0 \ldots b_{n-1})$ contains exactly two elements of B_{n+1}, namely the sets $\Delta(b_0 \ldots b_{n-1} 1)$ and $\Delta(b_0 \ldots b_{n-1} u)$. Now,

$$T^{n+1} \Delta(b_0 \ldots b_{n-1} 1) = [0, u - 1)$$

and

$$T^{n+1} \Delta(b_0 \ldots b_{n-1} u) = \big[0, T^{i+1}(u - 1) \big).$$

Similar arguments hold in case $T^n \Delta(b_0 \ldots b_{n-1}) = \big[0, T^i(\beta - u) \big)$, except that $T^i(u - 1)$ is replaced by $T^i(\beta - u)$.

For $n \geqq 1$, let $\bar{\kappa}(n)$ be the number of elements from B_n that contain two elements from B_{n+1}. Then clearly for all $n \geqq 1$,

$$(13) \qquad\qquad\qquad \kappa(n + 1) = \kappa(n) + \bar{\kappa}(n).$$

From the above we see that in order to get an upper bound on $\kappa(n)$, we need to study the behavior of the orbits of $u - 1$ and $\beta - u$. The following three lemmas say something about the first few elements of the orbits of these points. These lemmas are needed to guarantee that the total measure of the underlying space of the natural extension will be finite.

LEMMA 2.2. *If* $1 < \beta \leqq 2$ *and* $u < \beta$, *then* $u - 1 \notin \Delta(u)$.

PROOF. Since $\beta \leqq 2$, we have $1 - \frac{1}{\beta} \leqq \frac{1}{\beta}$. Thus $u\left(1 - \frac{1}{\beta}\right) \leqq \frac{u}{\beta} < 1$ and hence $u - 1 < \frac{u}{\beta}$. $\square$

Observe that B_n only contains fundamental intervals of which the first digit is either 1 or u. Let $\kappa_1(n)$ denote the number of elements $\Delta(b_0 \ldots b_{n-1})$ in B_n such that $b_0 = 1$ and $\kappa_2(n)$ the number of elements in B_n that have u as their first digit. Then of course for all $n \geqq 1$,

$$\kappa(n) = \kappa_1(n) + \kappa_2(n).$$

Let $\big\{ F(n) \big\}_{n \geqq 0}$ denote the Fibonacci sequence, i.e. let $F(0) = 0$, $F(1) = 1$ and for $n \geqq 2$, let $F(n) = F(n-1) + F(n-2)$. Lemma 2.2 implies that the number of elements of B_{n+1} would be maximal if the only elements of B_n that do not contain two elements from B_{n+1} are the elements $\Delta(b_0 \ldots b_{n-1})$ for which $T^n \Delta(b_0 \ldots b_{n-1}) = [0, u - 1)$. In this maximal situation we would have $\kappa_1(1) = \kappa_1(2) = 1$ and for $n \geqq 3$,

$$\kappa_1(n) = \kappa_1(n-1) + \kappa_1(n-2).$$

For κ_2 we would have that $\kappa_2(n) = \kappa_1(n+1)$. This means that under the conditions from Lemma 2.2, we have for all $n \geqq 1$ that $\kappa_1(n) \leqq F(n)$ and

$$(14) \qquad \kappa(n) = \kappa_1(n) + \kappa_2(n) \leqq F(n) + F(n+1) = F(n+2).$$

Let $G = \frac{1 + \sqrt{5}}{2}$ be the golden mean, i.e. the positive solution of the equation $x^2 - x - 1 = 0$.

LEMMA 2.3. *Let* $1 < \beta \leqq G$ *and* $u < \beta$. *Then* $u - 1, \beta - u \in \Delta(0)$.

PROOF. Since $\beta \leqq G$, we have $1 + \frac{1}{\beta} \geqq \beta$, so by equation (11), $u < \beta \leqq \left(1 + \frac{1}{\beta}\right)$. Thus $u - 1 < \frac{1}{\beta}$ and hence $u - 1 \in \Delta(0)$. On the other hand, since $\beta - \frac{1}{\beta} \leqq 1$, we have $\beta - u < \frac{1}{\beta}$ and thus $\beta - u \in \Delta(0)$. $\square$

REMARK 2.2. This lemma implies that for $1 < \beta \leqq G$ and for digit sets satisfying condition (11), we have $\kappa(2) = 2$. The largest amount of elements for B_n would be obtained if

$$\lambda\big(\left[0, T^i(u - 1)\right) \cap \Delta(u)\big) > 0 \quad \text{and} \quad \lambda\big(\left[0, T^i(\beta - u)\right) \cap \Delta(u)\big) > 0$$

for all odd values of i and thus $T^i(u-1), T^i(\beta-u) \in \Delta(0)$ for all even values of i. In this case,

$$\bar{\kappa}(n) = \begin{cases} \kappa(n), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

This would imply that for $n \geqq 1$, $\kappa(2n-1) = \kappa(2n) = 2^n$. In general we have that for all $n \geqq 1$, $\kappa(n) \leqq 2^{\lfloor n/2 \rfloor + 1}$.

LEMMA 2.4. *Let $m \geqq 2$ and $u < \beta$. If $1 < \beta \leqq 2^{\frac{1}{m}}$, then*

$$T^i(u-1), T^i(\beta-u) \in \Delta(0)$$

for all $i \in \{0, 1, \ldots, m-1\}$.

PROOF. The proof is by induction on m. Note that from Lemma 2.3 we know that $u - 1, \beta - u \in \Delta(0)$. Assume first that $m = 2$ and thus $\beta \leqq \sqrt{2} = 2^{\frac{1}{2}}$. Then

$$T(u-1) = \beta(u-1) \in \Delta(0) \quad \Leftrightarrow \quad \beta(u-1) < \frac{1}{\beta} \quad \Leftrightarrow \quad \frac{u}{\beta} < \frac{1}{\beta} + \frac{1}{\beta^3}.$$

If $\beta \leqq \sqrt{2}$, then $\frac{1}{\beta} + \frac{1}{\beta^3} > 1$ and since $u < \beta$, $T(u-1) \in \Delta(0)$. On the other hand, since $\beta - u \in \Delta(0)$, we have

$$T(\beta-u) = \beta(\beta-u) \in \Delta(0) \quad \Leftrightarrow \quad \beta^2 - \beta u < \frac{1}{\beta} \quad \Leftrightarrow \quad \beta - \frac{1}{\beta^2} < u.$$

If $\beta \leqq \sqrt{2}$, then $\beta - \frac{1}{\beta^2} < 1$, and thus $T(\beta-u) \in \Delta(0)$.

Now, assume that the result is true for some $k \geqq 2$. Let $m = k + 1$ and $\beta \leqq 2^{\frac{1}{k+1}}$. Then certainly $\beta \leqq 2^{\frac{1}{k}}$, so by induction $T^i(u-1), T^i(\beta-u) \in \Delta(0)$ for all $i \in \{0, 1, \ldots, k-1\}$. We only need to show that $T^k(u-1), T^k(\beta-u) \in \Delta(0)$. First consider $T^k(u-1) = \beta^k(u-1)$. We have

$$T^k(u-1) \in \Delta(0) \quad \Leftrightarrow \quad \beta^k(u-1) < \frac{1}{\beta} \quad \Leftrightarrow \quad \frac{u}{\beta} < \frac{1}{\beta} + \frac{1}{\beta^{k+2}}.$$

Since $\beta \leqq 2^{\frac{1}{k+1}}$, then

$$\frac{1}{\beta} + \frac{1}{\beta^{k+2}} \geqq \frac{3}{2 \cdot 2^{\frac{1}{k+1}}} \geqq \frac{3}{2\sqrt{2}} > 1.$$

Thus $T^k(u-1) \in \Delta(0)$. We now consider $T^k(\beta-u) = \beta^k(\beta-u)$ and see that

$$T^k(\beta-u) \in \Delta(0) \quad \Leftrightarrow \quad \beta^{k+1} - \beta^k u < \frac{1}{\beta} \quad \Leftrightarrow \quad \beta - \frac{1}{\beta^{k+1}} < u.$$

Since $\beta \leqq 2^{\frac{1}{k+1}}$, then

$$\beta - \frac{1}{\beta^{k+1}} \leqq 2^{\frac{1}{k+1}} - \frac{1}{2} \leqq \sqrt{2} - \frac{1}{2} < 1.$$

Thus $T^k(\beta - u) \in \Delta(0)$ and this proves the lemma. $\square$

REMARK 2.3. Suppose $2^{1/(m+1)} < \beta \leqq 2^{1/m}$ and $u < \beta$. Lemma 2.4 implies that $\kappa(i) = 2$ for $i \in \{1, \ldots, m\}$. By the same reasoning as in Remark 2.2, $\kappa(n)$ would obtain the largest possible value if

$$\lambda\big([0, T^i(u-1)) \cap \Delta(u)\big) > 0 \quad \text{and} \quad \lambda\big([0, T^i(\beta - u)) \cap \Delta(u)\big) > 0$$

for all $i = \ell m + (\ell - 1)$, $\ell \geqq 1$, and $T^i(u-1), T^i(\beta - u) \in \Delta(0)$ for all other values of i. This would imply

$$\bar{\kappa}(n) = \begin{cases} \kappa(n), & \text{if } n = \ell m + (\ell - 1) \text{ for some } \ell \geqq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, to get the maximal number of elements for B_n, we would have that if

$$(\ell - 1)m + \ell \leqq n \leqq \ell m + \ell$$

for some ℓ, then $\kappa(n) = 2^\ell$. So, in general we have that $\kappa(n) \leqq 2^{\lfloor \frac{n}{m} \rfloor + 1}$.

For all $n \geqq 1$, let D_n be the union of all full fundamental intervals of rank n, that are not a subset of any full fundamental interval of lower rank. From the next lemma it follows that the full fundamental intervals generate the Borel σ-algebra on $[0, 1)$.

LEMMA 2.5. We have

$$\lambda\left(\bigcup_{n \geqq 1} D_n\right) = \sum_{n \geqq 1} \lambda(D_n) = \lambda\big([0,1)\big) = 1.$$

PROOF. Notice that all of the sets D_n are disjoint. By (12) we have for each $n \geqq 1$, that

$$0 \leqq \lambda\left([0,1) \setminus \bigcup_{i=1}^n D_i\right) \leqq \frac{\kappa(n)}{\beta^n},$$

so it is enough to prove that $\lim_{n \to \infty} \frac{\kappa(n)}{\beta^n} = 0$. If $2 < \beta < 3$, then since $\kappa(n) \leqq 2^n$, we immediately have the result. For $1 < \beta \leqq G$, it follows from Remark 2.2 and Remark 2.3. Now, suppose $G < \beta \leqq 2$. Then by (14), we have that

$\kappa(n) \leqq F(n+2)$, where $F(n+2)$ is the $(n+2)$-th element of the Fibonacci sequence. For the elements of this sequence, there is a closed formula which gives

$$(15) \qquad\qquad F(n) = \frac{G^n - (1-G)^n}{\sqrt{5}}.$$

So

$$\frac{\kappa(n)}{\beta^n} \leqq \frac{1}{\sqrt{5}} \left[G^2 \left(\frac{G}{\beta} \right)^n - (1-G)^2 \left(\frac{1-G}{\beta} \right)^n \right].$$

Since $G < \beta \leqq 2$, also in this case $\lim_{n \to \infty} \frac{\kappa(n)}{\beta^n} = 0$ and this proves the lemma.
$\square$

REMARK 2.4. The fact that $\Delta(0)$ is a full fundamental interval of rank 1 allows us to construct full fundamental intervals of arbitrary small Lebesgue measure. This, together with the previous lemma, guarantees that we can write each subinterval of $[0, 1)$ as a countable union of full fundamental intervals. Thus, the full fundamental intervals generate the Borel σ-algebra on $[0, 1)$.

Notice that for the cases illustrated by Figure 1(b) and 1(c), we can define the partitions $\Delta^{(n)}$, the sets B_n and the numbers $\kappa(n)$ in a similar way. The only differences are that the support of the ACIM is given by the interval $[0, u-1)$ and that $\Delta(1)$ is the only full fundamental interval of rank 1. In that sense, $\Delta(1)$ plays the role of $\Delta(0)$ above. Since in these cases we have $2 < \beta < 3$ and since $\kappa(n) \leqq 2^n$ for all $n \geqq 1$, we can prove a lemma similar to Lemma 2.5, i.e. we can prove that the full fundamental intervals generate the Borel σ-algebra on the support of the ACIM.

3. A natural extension of T

For the version of the natural extension, we will define a space R, using the element of B_n. For $n \geqq 1$, define the collections

$$R_n = \left\{ T^n \Delta(b_0 \ldots b_{n-1}) \times \left[0, \frac{1}{\beta^n} \right) : \Delta(b_0 \ldots b_{n-1}) \in B_n \right\}.$$

So to each element of B_n, there corresponds an element of R_n and thus the number of elements in R_n is given by $\kappa(n)$. We enumerate the elements of R_n and write $R_n = \left\{ R_{(n,i)} : 1 \leqq i \leqq \kappa(n) \right\}$. Thus, for each $\Delta(b_0 \ldots b_{n-1})$ $\in B_n$ there exists a unique $1 \leqq i \leqq \kappa(n)$ such that $T^n \Delta(b_0 \ldots b_{n-1}) \times \left[0, \frac{1}{\beta^n} \right)$

$= R_{(n,i)}$. Let $R_0 = [0, 1) \times [0, 1)$ and let R be the disjoint union of all these sets, i.e.

$$R = R_0 \times \{0\} \times \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\kappa(n)} R_{(n,i)} \times \{n\} \times \{i\}.$$

The σ-algebra $\mathcal{R}$ on R is the disjoint union of the Borel σ-algebras on all the rectangles $R_{(n,i)}$ and R_0. Let λ_R be the measure on R, given by the two dimensional Lebesgue measure on each of these rectangles. Define the probability measure ν on R by setting $\nu(E) = \frac{1}{\lambda_R(R)} \lambda_R(E)$ for all $E \in \mathcal{R}$. The next lemma says that this measure is well defined and finite.

LEMMA 3.1. $\lambda_R(R) < \infty$.

PROOF. Set $\kappa(0) = 1$. Then,

$$\lambda_R(R) \leqq \sum_{n=0}^{\infty} \frac{\kappa(n)}{\beta^n}.$$

Using the same arguments as in the proof of Lemma 2.5, we can show that the sum on the right hand side converges for all $1 < \beta < 3$. $\quad\square$

The transformation $\mathcal{T} : R \to R$ is defined piecewise on each rectangle. If $(x, y) \in R_0$, then

$$\mathcal{T}(x, y, 0, 0) = \begin{cases} \left(Tx, \dfrac{y}{\beta}, 0, 0\right), & \text{if } x \in \Delta(0), \\[2mm] \left(Tx, \dfrac{y}{\beta}, 1, 1\right), & \text{if } x \in \Delta(1), \\[2mm] \left(Tx, \dfrac{y}{\beta}, 1, 2\right), & \text{if } x \in \Delta(u). \end{cases}$$

For each $n \geqq 1$, $1 \leqq i \leqq \kappa(n)$, if

$$R_{(n,i)} = T^n \Delta(b_0 \ldots b_{n-1}) \times \left[0, \frac{1}{\beta^n}\right),$$

then $\mathcal{T}$ maps this rectangle to the rectangles corresponding to the fundamental intervals contained in $\Delta(b_0 \ldots b_{n-1})$ in the following way. If $\Delta(b_0 \ldots b_{n-1}0)$ is full and $(x, y) \in R_{(n,i)}$ with $x \in \Delta(0)$, then

$$\mathcal{T}(x, y, n, i) = \left(Tx, \frac{b_0}{\beta} + \frac{b_1}{\beta^2} + \cdots + \frac{b_{n-1}}{\beta^n} + \frac{y}{\beta}, 0, 0\right).$$

If $\Delta(b_0 \ldots b_{n-1}b_n) \in B_{n+1}$ and j is the index of the corresponding set in R_{n+1}, then for $(x,y) \in R_{(n,i)}$ with $x \in \Delta(b_n)$, we set

$$\mathcal{T}(x,y,n,i) = \left(Tx, \frac{y}{\beta}, n+1, j\right).$$

In Fig. 2 we show the different situations in case

$$T^n \Delta(b_0 \ldots b_{n-1}) = \left[0, T^k(\beta - u)\right)$$

for some $k < n$. If $T^n \Delta(b_0 \ldots b_{n-1}) = \left[0, T^k(u-1)\right)$ for some $k < n$, the pictures look exactly the same with $T^k(u-1)$ in place of $T^k(\beta - u)$.
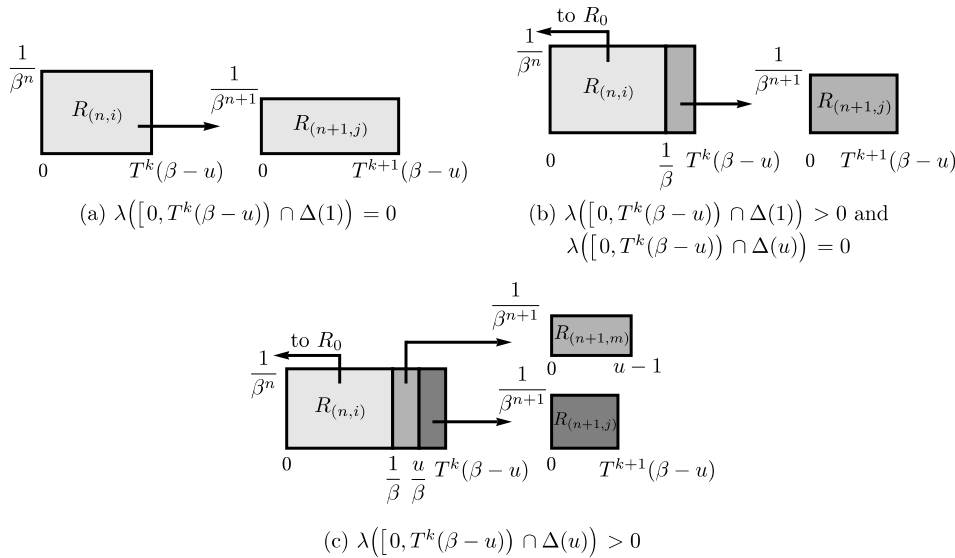

(a) $\lambda\left(\left[0, T^k(\beta - u)\right) \cap \Delta(1)\right) = 0$

(b) $\lambda\left(\left[0, T^k(\beta - u)\right) \cap \Delta(1)\right) > 0$ and $\lambda\left(\left[0, T^k(\beta - u)\right) \cap \Delta(u)\right) = 0$

(c) $\lambda\left(\left[0, T^k(\beta - u)\right) \cap \Delta(u)\right) > 0$

Fig. 2: *The arrows indicate the action of $\mathcal{T}$ in case* $T^n \Delta(b_0 \ldots b_{n-1}) = \left[0, T^k(\beta - u)\right)$, *for some* $k < n$

If a rectangle $R_{(n,i)}$ corresponds to a fundamental interval $\Delta(b_0 \ldots b_{n-1})$ such that $\Delta(b_0 \ldots b_{n-1}0)$ is non-full, then this is the only fundamental interval contained in $\Delta(b_0 \ldots b_{n-1})$. $\mathcal{T}$ then maps the rectangle $R_{(n,i)}$ bijectively to the rectangle $R_{(n+1,j)}$, corresponding to $\Delta(b_0 \ldots b_{n-1}0)$. Otherwise, $\Delta(b_0 \ldots b_{n-1}0)$ is a full fundamental interval contained in $\Delta(b_0 \ldots b_{n-1})$ and the rectangle $R_{(n,i)}$ is partly mapped surjectively onto some rectangle $R_{(n+1,j)}$ and partly into R_0. From Lemma 2.5 it now follows that $\mathcal{T}$ is bijective.

Let $\pi \colon R \to [0,1)$ be the projection onto the first coordinate. Define the measure μ on $\left([0,1), \mathcal{B}([0,1))\right)$ by pulling back the measure ν, i.e.

for all measurable sets $E \in \mathcal{B}\big([0,1)\big)$, let $\mu(E) = \nu(\pi^{-1}E)$. In order to show that $(R, \mathcal{R}, \nu, \mathcal{T})$ is a version of the natural extension of the system $\big([0,1), \mathcal{B}\big([0,1)\big), \mu, T\big)$ with π as the factor map, we will prove all of the following:

(i) π is a surjective, measurable and measure preserving map from R to $[0,1)$.

(ii) For all $x \in R$, we have $(T \circ \pi)(x) = (\pi \circ \mathcal{T})(x)$.

(iii) $\mathcal{T} : R \to R$ is an invertible transformation.

(iv) $\mathcal{R} = \bigvee_{n=0}^{\infty} \mathcal{T}^n \pi^{-1}\big(\mathcal{B}\big([0,1)\big)\big)$, where $\bigvee_{n=0}^{\infty} \mathcal{T}^n \pi^{-1}\big(\mathcal{B}\big([0,1)\big)\big)$ is the smallest σ-algebra containing the σ-algebras $\mathcal{T}^n \pi^{-1}\big(\mathcal{B}\big([0,1)\big)\big)$ for all $n \geqq 1$.

It is clear that π is surjective and measurable and that $T \circ \pi = \pi \circ \mathcal{T}$. Since $\mathcal{T}$ expands by a factor β in the first coordinate and contracts by a factor β in the second coordinate, it is also clear that $\mathcal{T}$ is invariant with respect to the measure ν. Then $\mu = \nu \circ \pi^{-1}$ defines a T-invariant probability measure on $\big([0,1), \mathcal{B}\big([0,1)\big)\big)$, that is equivalent to the Lebesgue measure on $[0,1)$ and π is a measure preserving map. This shows (i) and (ii). The invertibility of $\mathcal{T}$ follows from Remark 2.4, so that leaves only (iv). To prove (iv) we will have a look at the structure of the fundamental intervals and we will introduce some more notation.

Let $\Delta(b_0 \ldots b_{n-1})$ be a fundamental interval. We can divide the block of digits $b_0 \ldots b_{n-1}$ into M subblocks, $C_1, \ldots, C_M$, for some $M \geqq 1$, where each subblock C_i, $1 \leqq i \leqq M-1$, corresponds to a full fundamental interval. The last subblock, C_M, corresponds to a full fundamental interval exactly when $\Delta(b_0 \ldots b_{n-1})$ is full. We can make this precise, using the notion of return times to R_0. For points $(x,y) \in R_0$ define the *first return time to R_0* by

$$r_1(x,y) = \inf \big\{ n \geqq 1 : \mathcal{T}^n(x, y, 0, 0) \in R_0 \times \{0\} \times \{0\} \big\}$$

and for $k \geqq 1$, let the *k-th return time to R_0* be given recursively by

$$r_k(x,y) = \inf \big\{ n > r_{k-1}(x,y) : \mathcal{T}^n(x, y, 0, 0) \in R_0 \times \{0\} \times \{0\} \big\}.$$

By the Poincaré Recurrence Theorem, we have $r_k(x,y) < \infty$ for almost all $(x,y) \in R_0$. Notice that this notion of return time depends only on x, i.e. for all $y, y' \in R_0$ and all $k \geqq 1$, $r_k(x,y) = r_k(x,y')$. So we can write $r_k(x)$ instead of $r_k(x,y)$. If $\Delta(b_0 \ldots b_{n-1}) \in \Delta^{(n)}$, then for all $m \leqq n$, $\mathcal{T}^m$ maps the whole rectangle $\Delta(b_0 \ldots b_{n-1}) \times [0,1) \subseteq R_0$ to the same rectangle in R. So up to a certain $\ell \leqq n$, the i-th return time to R_0 is equal for all elements in $\Delta(b_0 \ldots b_{n-1})$. Now suppose that $\Delta(b_0 \ldots b_{n-1}) \in \Delta^{(n)}$ is a full fundamental interval, then there is an $M \geqq 1$ and there are numbers r_i, $1 \leqq i \leqq M$ such

that $r_i = r_i(x)$ for all $x \in \Delta(b_0 \ldots b_{n-1})$ and $r_M = n$. Put $r_0 = 0$. We can also obtain the numbers r_i inductively as follows. Let

$$r_1 = \inf \left\{ j > 0 : T^{j+1} \Delta(b_0 \ldots b_j) = [0, 1) \right\}$$

and if $r_1, \ldots, r_{k-1}$ are already known, let

$$r_k = \inf \left\{ j > r_{k-1} : T^{j+1} \Delta(b_{r_{k-1}} \ldots b_j) = [0, 1) \right\}.$$

Take for $1 \leqq i \leqq M$,

(16) $$C_i = b_{r_{i-1}} \ldots b_{r_i-1}.$$

Let $|C_i|$ denote the number of digits of the block C_i. The blocks have the following properties:

(p1) For $1 \leqq i \leqq M$, $|C_i| = r_i - r_{i-1}$.

(p2) If $b_{r_i} = 0$, then $r_{i+1} = r_i + 1$. This means that if a subblock begins with the digit 0, then 0 is the only digit in this subblock. So, C_{i+1} consists just of the digit 0.

(p3) For all $i \in \{1, \ldots, M\}$, $\Delta(C_i)$ is a full fundamental interval of rank $|C_i|$.

The next lemma is the last step in proving that $(R, \mathcal{R}, \nu, \mathcal{T})$ is the natural extension of the space $\big([0, 1), \mathcal{B}\big([0, 1)\big), \mu, T\big)$.

LEMMA 3.2. *Let* $(R, \mathcal{R}, \nu, \mathcal{T})$ *and* $\big([0, 1), \mathcal{B}\big([0, 1)\big), \mu, T\big)$ *be the dynamical systems defined above. Then*

$$\mathcal{R} = \bigvee_{n=0}^{\infty} \mathcal{T}^n\big(\pi^{-1}\big(\mathcal{B}\big([0, 1)\big)\big)\big).$$

PROOF. It is clear that

$$\bigvee_{n=0}^{\infty} \mathcal{T}^n\big(\pi^{-1}\big(\mathcal{B}\big([0, 1)\big)\big)\big) \subseteqq \mathcal{R}.$$

By Lemma 2.5 we know that the direct products of the full fundamental intervals contained in the rectangle R_0 generate the Borel σ-algebra on this rectangle. The same holds for all the rectangles $R_{(n,i)}$. First, let

$$\Delta(d_0 \ldots d_{p-1}) \times \Delta(e_0 \ldots e_{q-1})$$

be a generating rectangle in R_0, where $\Delta(d_0 \ldots d_{p-1})$ and $\Delta(e_0 \ldots e_{q-1})$ are full fundamental intervals. For the set $\Delta(e_0 \ldots e_{q-1})$ construct the subblocks

$C_1, \ldots, C_M$ as in (16). By property (p3) and Lemma 2.1, $\Delta(C_M C_{M-1} \ldots C_1 d_0 \ldots d_{p-1})$ is a full fundamental interval of rank $p+q$. Then

$$\pi^{-1}\big(\Delta(C_M C_{M-1} \ldots C_1 d_0 \ldots d_{p-1})\big) \cap \big(R_0 \times \{0\} \times \{0\}\big)$$

$$= \Delta(C_M C_{M-1} \ldots C_1 d_0 \ldots d_{p-1}) \times [0,1) \times \{0\} \times \{0\}.$$

Since $\Delta(e_0 \ldots e_{q-1})$ is a full fundamental interval, it can be proven by induction that for all $i \in \{1, \ldots, q-1\}$, $T^i \Delta(e_0 \ldots e_{q-1}) = \Delta(e_i \ldots e_{q-1})$. This, together with the definitions of the blocks C_i and the transformation $\mathcal{T}$ leads to

$$\mathcal{T}^q\big(\pi^{-1}(C_M C_{M-1} \ldots C_1 d_0 \ldots d_{p-1})\big) \cap \big(R_0 \times \{0\} \times \{0\}\big)$$

$$= \Delta(d_0 \ldots d_{p-1}) \times \Delta(C_1 C_2 \ldots C_M) \times \{0\} \times \{0\}.$$

So

$$\Delta(d_0 \ldots d_{p-1}) \times \Delta(e_0 \ldots e_{q-1}) \times \{0\} \times \{0\} \subseteq \bigvee_{n=0}^{\infty} \mathcal{T}^n \pi^{-1}\big(\mathcal{B}\big([0,1)\big)\big).$$

Now, for $n \geqq 1$ and $i \in \big\{1, 2, \ldots, \kappa(n)\big\}$, let $R_{(n,i)}$ be a rectangle in R_n and suppose that it corresponds to the fundamental interval $\Delta(b_0 \ldots b_{n-1}) \in B_n$. Hence,

$$R_{(n,i)} = T^n \Delta(b_0 \ldots b_{n-1}) \times \left[0, \frac{1}{\beta^n}\right) \times \{n\} \times \{i\}.$$

Let $\Delta(d_0 \ldots d_{p-1}) \times \Delta(e_0 \ldots e_{q-1}) \times \{n\} \times \{i\}$ be a generating rectangle for the Borel σ-algebra on the rectangle $R_{(n,i)}$. So $\Delta(d_0 \ldots d_{p-1})$ and $\Delta(e_0 \ldots e_{q-1})$ are again full fundamental intervals. Notice that

$$\Delta(e_0 \ldots e_{q-1}) \subseteq \Delta(\underbrace{0 \ldots 0}_{n \text{ times}}),$$

which means that $q \geqq n$. Also, for all $i \in \{0, \ldots, n-1\}$, $e_i = 0$ and thus $r_{i+1} = i+1$. So, if we divide $e_0 \ldots e_{q-1}$ into subblocks C_i as before, we get that $C_1 = C_2 = \cdots = C_n = 0$, that $M \geqq n$ and that $|C_{n+1}| + \cdots + |C_M| = q - n$. Consider the set

$$C = \Delta(C_M C_{M-1} \ldots C_{n+1} b_0 \ldots b_{n-1} d_0 \ldots d_{p-1}).$$

We will show the following.

Claim: The set C is a fundamental interval of rank $p+q$ and $T^q C = \Delta(d_0 \ldots d_{p-1})$.

First notice that

$$C = \Delta(C_M C_{M-1} \ldots C_{n+1}) \cap T^{n-q} \Delta(b_0 \ldots b_{n-1}) \cap T^{-q} \Delta(d_0 \ldots d_{p-1}).$$

So obviously,

$$T^q C \subseteqq T^q \Delta(C_M C_{M-1} \ldots C_{n+1}) \cap T^n \Delta(b_0 \ldots b_{n-1}) \cap \Delta(d_0 \ldots d_{p-1}).$$

By Lemma 2.1, $\Delta(C_M C_{M-1} \ldots C_{n+1})$ is a full fundamental interval of rank $q - n$, so $T^q \Delta(C_M C_{M-1} \ldots C_{n+1}) = [0, 1)$. Now, by the definition of $R_{(n,i)}$ we have that

$$(17) \qquad\qquad \Delta(d_0 \ldots d_{p-1}) \subseteqq T^n \Delta(b_0 \ldots b_{n-1}),$$

and thus $T^q C \subseteqq \Delta(d_0 \ldots d_{p-1})$.

For the other inclusion, let $z \in \Delta(d_0 \ldots d_{p-1})$. By (17), there is an element y in $\Delta(b_0 \ldots b_{n-1})$, such that $T^n y = z$. And since $T^{q-n} \Delta(C_M C_{M-1} \ldots C_{n+1})$ $= [0, 1)$, there is an $x \in \Delta(C_M C_{M-1} \ldots C_{n+1})$ with $T^{q-n} x = y$, so $T^q x = z$. This means that

$$z \in T^q \Delta(C_M C_{M-1} \ldots C_{n+1}) \cap T^n \Delta(b_0 \ldots b_{n-1}) \cap \Delta(d_0 \ldots d_{p-1}).$$

So $T^q C = \Delta(d_0 \ldots d_{p-1})$ and this proves the claim.

Consider the set $D = \pi^{-1}(C) \cap \left(R_0 \times \{0\} \times \{0\} \right)$. Then as before, we have

$$\mathcal{T}^{q-n} D = \Delta(b_0 \ldots b_{n-1} d_0 \ldots d_{p-1}) \times \Delta(C_{n+1} C_{n+2} \ldots C_M) \times \{0\} \times \{0\}.$$

And after n more steps,

$$\mathcal{T}^q D = \Delta(d_0 \ldots d_{p-1}) \times \Delta(\underbrace{00 \ldots 0}_{n \text{ times}} C_{n+1} \ldots C_M) \times \{n\} \times \{i\}$$

$$= \Delta(d_0 \ldots d_{p-1}) \times \Delta(e_0 \ldots e_{q-1}) \times \{n\} \times \{i\}.$$

So,

$$\Delta(d_0 \ldots d_{p-1}) \times \Delta(e_0 \ldots e_{q-1}) \times \{n\} \times \{i\} \in \bigvee_{n=0}^{\infty} \mathcal{T}^n \pi^{-1} \big(\mathcal{B}([0, 1)) \big)$$

and thus we see that

$$\mathcal{R} = \bigvee_{n=0}^{\infty} \mathcal{T}^n \pi^{-1} \big(\mathcal{B}([0, 1)) \big). \quad \square$$

This gives the following theorem.

THEOREM 3.1. *The dynamical system $(R, \mathcal{R}, \nu, \mathcal{T})$ is a version of the natural extension of the dynamical system $\big([0,1), \mathcal{B}\big([0,1)\big), \mu, T\big)$. Here, the measure μ, given by $\mu(E) = \nu\big(\pi^{-1}(E)\big)$ for all measurable sets E, is the ACIM of T and the density of μ is equal to the density from equation* (10).

PROOF. The fact that $(R, \mathcal{R}, \nu, \mathcal{T})$ is a version of the natural extension of the system $\big([0,1), \mathcal{B}\big([0,1)\big), \mu, T\big)$ follows from Remark 2.4, the properties of the map π and Lemma 3.2. Now for each measurable set $E \in \mathcal{B}\big([0,1)\big)$, we have

$$\mu(E) = \nu\big(\pi^{-1}(E)\big)$$

$$= \frac{1}{\lambda_R(R)}\left[\lambda(E) + \sum_{n=1}^{\infty} \sum_{\Delta(b_0 \ldots b_{n-1}) \in B_n} \frac{1}{\beta^n} \lambda\big(E \cap T^n \Delta(b_0 \ldots b_{n-1})\big)\right]$$

$$= \frac{1}{\lambda_R(R)} \int_E \left[1 + \sum_{n=1}^{\infty} \sum_{\Delta(b_0 \ldots b_{n-1}) \in B_n} \frac{1}{\beta^n} \, 1_{T^n \Delta(b_0 \ldots b_{n-1})}\right] d\lambda.$$

Here $T^n \Delta(b_0 \ldots b_{n-1})$ has the form

$$\big[0, T^i(u-1)\big) \text{ or } \big[0, T^i(\beta - u)\big)$$

for some $0 \leqq i < n$. So, the density of the measure μ equals the density from equation (10).   □

REMARK 3.1. (i) The above definitions of the space R and the transformation $\mathcal{T}$ can be adapted quite easily for the cases illustrated by Figure 1(b) and 1(c). We let $\Delta(1)$ take the role of $\Delta(0)$ and consider the orbits of the points 1 and $\beta(u-1) - u$. In general, the sets B_n will contain more elements, but since $2 < \beta < 3$, it is immediate that $\lambda_R(R) < \infty$. This shows that we can construct a version of the natural extension of T, also for these two cases.

(ii) Let R_0' be the set obtained from R_0 by removing the set of measure zero of elements which do not return to R_0, i.e. we remove those (x, y) for which $r_1(x, y) = \infty$. Let $\mathcal{W} : R_0' \to R_0'$ be the transformation induced by $\mathcal{T}$, i.e. for all $(x, y) \in R_0'$, let

$$\mathcal{W}(x, y) = \mathcal{T}^{r_1(x,y)}(x, y, 0, 0).$$

Then the dynamical system $(R_0', \mathcal{B}(R_0'), \lambda \times \lambda, \mathcal{W})$, where $\mathcal{B}(R_0')$ is the Borel σ-algebra on R_0', is isomorphic to the natural extension of a GLS-transformation as defined in [4]. This implies that the system $\big(R_0', \mathcal{B}(R_0'), \lambda \times \lambda, \mathcal{W}\big)$ is Bernoulli.

Using this invariant measure, we will show that T is an exact transformation. Since the full fundamental intervals generate the Borel σ-algebra on the support of the ACIM, by a result of Rohlin [15], it is enough to show that there exists a universal constant $\gamma > 0$, such that for any full fundamental interval $\Delta(b_0 \ldots b_{n-1})$ and any measurable subset $E \subseteqq \Delta(b_0 \ldots b_{n-1})$, we have

$$\mu(T^n E) \leqq \gamma \cdot \frac{\mu(E)}{\mu\big(\Delta(b_0 \ldots b_{n-1})\big)}.$$

To this end, define two constants, $c_1, c_2 > 0$, by

$$c_1 = \frac{1}{\lambda_R(R)} \quad \text{and} \quad c_2 = 1 + \sum_{n=1}^{\infty} \sum_{\Delta(b_0 \ldots b_{n-1}) \in B_n} \frac{1}{\beta^n}.$$

Then for all measurable sets E, we have

$$c_1 \lambda(E) \leqq \mu(E) \leqq c_1 c_2 \lambda(E).$$

Now, let $\Delta(b_0 \ldots b_{n-1})$ be a full fundamental interval of rank n. Then by (8),

$$\lambda\big(\Delta(b_0 \ldots b_{n-1})\big) = \frac{1}{\beta^n}.$$

Let $E \subseteqq \Delta(b_0 \ldots b_{n-1})$ be a measurable set. Then

$$\lambda(T^n E) = \beta^n \lambda(E) = \frac{1}{\lambda\big(\Delta(b_0 \ldots b_{n-1})\big)} \lambda(E).$$

Now,

$$\mu(T^n E) \leqq c_1 c_2 \lambda(T^n E) = c_1 c_2 \frac{\lambda(E)}{\lambda\big(\Delta(b_0 \ldots b_{n-1})\big)}$$

$$\leqq c_1 c_2 \frac{\mu(E) c_1 c_2}{c_1 \mu\big(\Delta(b_0 \ldots b_{n-1})\big)} = c_1 c_2^2 \frac{\mu(E)}{\mu\big(\Delta(b_0 \ldots b_{n-1})\big)}.$$

If we take $\gamma = c_1 c_2^2$, then $\gamma > 0$ and

$$\mu(T^n E) \leqq \gamma \cdot \frac{\mu(E)}{\mu\big(\Delta(b_0 \ldots b_{n-1})\big)}.$$

Thus, T is exact and hence mixing of all orders. Furthermore, the natural extension $\mathcal{T}$ is a K-automorphism. By a result of Rychlik [16], it follows immediately that T is weakly Bernoulli.

4. A specific example

In this section we will consider one specific example of a β-transformation with three digits. Let G be the golden mean as before and take $\beta = G$. Consider the allowable digit set $A = \left\{0, 1, \frac{4}{3}\right\}$. The partition $\Delta = \left\{\Delta(0), \Delta(1), \Delta\left(\frac{4}{3}\right)\right\}$ is given by

$$\Delta(0) = \left[0, \frac{1}{\beta}\right), \quad \Delta(1) = \left[\frac{1}{\beta}, \frac{4}{3\beta}\right), \quad \Delta\left(\frac{4}{3}\right) = \left[\frac{4}{3\beta}, 1\right)$$

and the transformation is then $Tx = \beta x - j$, if $x \in \Delta(j)$.

The points $x = \frac{1}{3}$ and $x = \beta - \frac{4}{3}$ are of special interest and their orbits under T are as follows:

$$\frac{1}{3}, \quad T\frac{1}{3} = \frac{\beta}{3}, \quad T^2\frac{1}{3} = \frac{\beta^2}{3}, \quad T^3\frac{1}{3} = \frac{1}{3\beta^3},$$

$$T^4\frac{1}{3} = \frac{1}{3\beta^2}, \quad T^5\frac{1}{3} = \frac{1}{3\beta}, \quad T^6\frac{1}{3} = \frac{1}{3},$$

$$\beta - \frac{4}{3}, \quad T\left(\beta - \frac{4}{3}\right) = 1 - \frac{\beta}{3}, \quad T^2\left(\beta - \frac{4}{3}\right) = \frac{2}{3}\beta - \frac{1}{3}, \quad T^3\left(\beta - \frac{4}{3}\right) = \frac{1}{3\beta}.$$
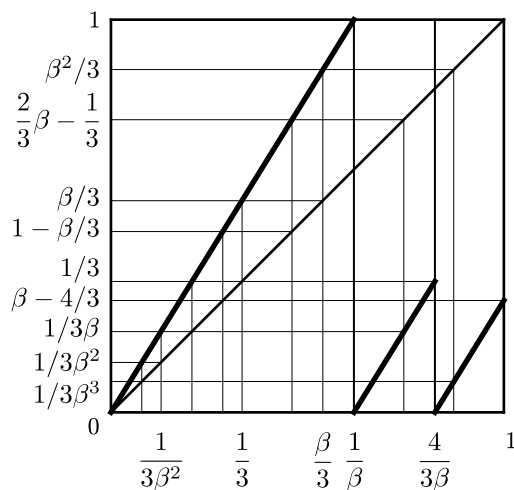

Fig. 3: The orbits of $\frac{1}{3}$ and $\beta - \frac{4}{3}$ under the greedy β-transformation with $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \left\{0, 1, \frac{4}{3}\right\}$

Thus their greedy β-expansions are given by

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{d_n^{(1)}}{\beta^n} =_\beta \left(00\frac{4}{3}00 \right)^\omega, \quad \beta - \frac{4}{3} = \sum_{n=1}^{\infty} \frac{d_n^{(2)}}{\beta^n} =_\beta 001 \left(0000\frac{4}{3} \right)^\omega.$$

Notice that $\beta - \frac{4}{3}$ would be the image of 1 under T, if T were not restricted to the half open interval $[0,1)$. Fig. 3 shows the orbits of both points under T.

For each non-full fundamental interval, $\Delta(b_0 \ldots b_{n-1})$, the set $T^n \Delta(b_0 \ldots b_{n-1})$ is one of the following:

$$\left[0, \beta - \frac{4}{3} \right), \quad \left[0, 1 - \frac{\beta}{3} \right), \quad \left[0, \frac{2}{3}\beta - \frac{1}{3} \right), \quad \left[0, \frac{1}{3} \right), \quad \left[0, \frac{\beta}{3} \right),$$

$$\left[0, \frac{\beta^2}{3} \right), \quad \left[0, \frac{1}{3\beta^3} \right), \quad \left[0, \frac{1}{3\beta^2} \right), \quad \left[0, \frac{1}{\beta} \right).$$

This means that we could give all the elements of R_n explicitly. We will not do this, since Fig. 4 speaks for itself. The space R contains all the rectangles shown in the figure.
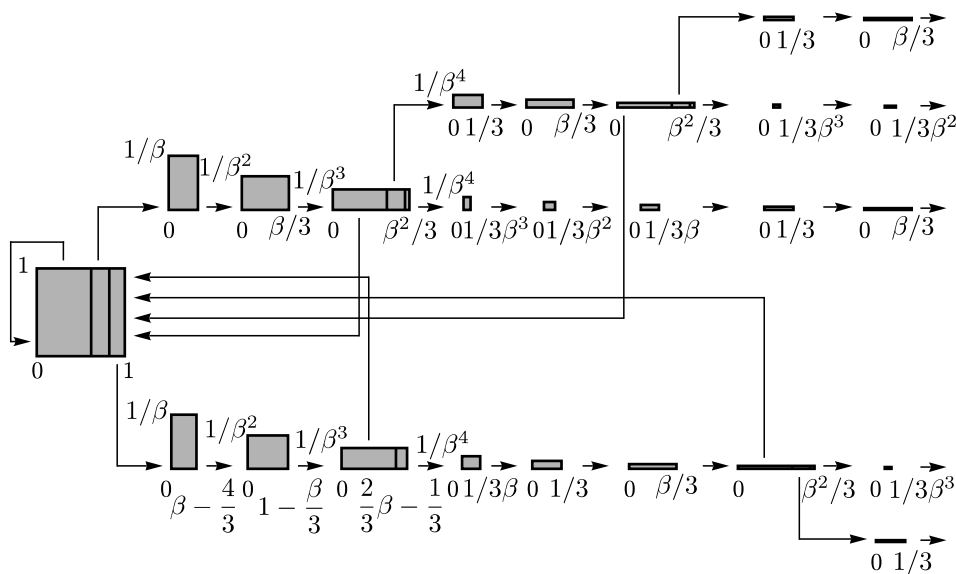

Fig. 4: The space R consists of all these rectangles. The arrows indicate where the rectangles are mapped under T

To determine the invariant measure of T, we need to project the Lebesgue measure on the rectangles of R onto the first coordinate. Therefore, we need to add up the heights of all the rectangles which have the same interval in the first coordinate. To do this, we only need to determine the total height of the rectangles of the form $\left[0, \frac{1}{3}\right) \times \left[0, \frac{1}{\beta^n}\right)$, since the total height of all the other rectangles in R can be deduced from this. These heights can be found by using the Fibonacci numbers $F(n)$, as defined before. Then $\kappa(1) = 2$ and observe that for $n \geqq 2$, $\kappa(n) = F\left(\left\lfloor \frac{n-1}{3} \right\rfloor + 2\right) + F\left(\left\lfloor \frac{n-2}{3} \right\rfloor + 1\right)$. For $n = 3k+1$, $k \geqq 0$, the number of rectangles in R of the form $\left[0, \frac{1}{3}\right) \times \left[0, \frac{1}{\beta^n}\right)$ is equal to $F(k+1)$ and for $n = 3k+2$, $k \geqq 1$, this number is equal to $F(k)$. Formula (15) gives that the total height of all these rectangle is equal to

$$
\sum_{k=0}^{\infty} \frac{F(k+1)}{\beta^{3k+1}} + \sum_{k=1}^{\infty} \frac{F(k)}{\beta^{3k+2}} = \frac{1}{\beta}\left[1 + \sum_{k=1}^{\infty} \frac{\beta F(k+1) + F(k)}{\beta^{3k+1}}\right]
$$

$$
= \frac{1}{\beta}\left[1 + \frac{1}{\sqrt{5}}\left(\beta \sum_{k=1}^{\infty} \frac{1}{\beta^{2k}} + \frac{1}{\beta}\sum_{k=1}^{\infty} \frac{1}{\beta^{2k}}\right)\right]
$$

$$
= \frac{1}{\beta}\left[1 + \frac{1}{\sqrt{5}}\left(\left(\beta + \frac{1}{\beta}\right)\left(\frac{\beta^2}{\beta^2-1} - 1\right)\right)\right] = \frac{1}{\beta}\left[1 + \frac{1}{\sqrt{5}}(3 - \beta)\right] = \frac{1}{3}.
$$

The total height of the rectangles of the form $\left[0, \frac{\beta}{3}\right) \times \left[0, \frac{1}{\beta^n}\right)$ is now equal to $\frac{1}{\beta}$, that of the rectangles of the form $[0, \frac{\beta^2}{3}) \times \left[0, \frac{1}{\beta^n}\right)$ is $\frac{1}{\beta^2}$, etc. The total height of the rectangles $\left[0, \frac{1}{3\beta}\right) \times \left[0, \frac{1}{\beta^n}\right)$ is given by $\frac{1}{\beta^4} + \frac{1}{\beta^5} = \frac{1}{\beta^3}$. Then the density function of the invariant probability measure of T, $h : [0,1) \to [0,1)$, equivalent to the Lebesgue measure on $[0,1)$, is given by

$$
h(x) = \frac{3}{58 - 31\beta}\left[\frac{1}{\beta}\,1_{[0,\beta-\frac{4}{3})}(x) + \frac{1}{\beta^2}\,1_{[0,1-\frac{\beta}{3})}(x) + \frac{1}{\beta^3}\,1_{[0,\frac{2}{3}\beta-\frac{1}{3})}(x)\right.
$$

$$
+ 1_{[0,\frac{1}{3})}(x) + \frac{1}{\beta}\,1_{[0,\frac{\beta}{3})}(x) + \frac{1}{\beta^2}\,1_{[0,\frac{\beta^2}{3})}(x) + \frac{1}{\beta^3}\,1_{[0,\frac{1}{3\beta^3})}(x)
$$

$$
\left. + \frac{1}{\beta^4}\,1_{[0,\frac{1}{3\beta^2})}(x) + \frac{1}{\beta^3}1_{[0,\frac{1}{3\beta})}(x) + 1_{[0,1)}(x)\right].
$$

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